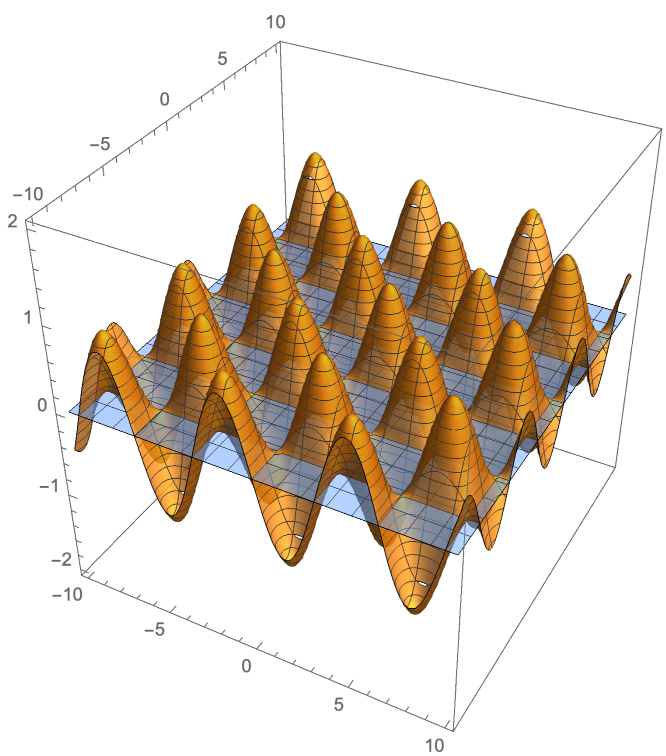
Pre-Calculus

Everything you need to know in less than 200 pages

AIDAN WITECK



Introduction

Before I begin discussing any part of this book, I need to thank a few people. First, Dr. Catherine Scott for being my wonderful Pre-Calculus teacher; thank you - I couldn't do any of this without you. Next, I'd like to thank my physics teacher, Dr. John Dell, for giving me a much deeper insight into math and the world around me than I ever thought possible, and for consistently making me laugh... IN SCHOOL (one of the hardest things to do). Enjoy your well-deserved retirement. I'd also like to thank my family for providing me with much-needed support, and Ms. Myra Spoden for providing me with guidance throughout the writing and publishing process. Lastly, I'd like to thank my good friend Stephen Arndt for tutoring me in my freshman year when I was struggling. He is seriously one of the kindest, most selfless people I have ever met.

One challenge I had throughout my first semester at Thomas Jefferson High School for Science and Technology ("TJ") was difficulty adjusting, especially with math. I was used to half-listening to my teacher's lectures, cramming for 30 minutes the night before tests, and then doing well on them. After a very short time at TJ, however, I quickly realized the results I was accustomed to would not be the case if I continued with my same approach and level of effort. Although I tried to study more effectively and efficiently, including doing all the review sheets I was given, I performed below my standards on basically every test. I attribute this to three main things: bad study habits, a lack of useful resource materials to help support success in rigorous math courses, and not taking summer-school classes before my freshmen year to prepare me better for what was to come.

The purpose of this book is to solve two of these issues. First, I have "attempted" to provide a deeper analysis of topics covered in Pre-Calculus, designing this book to be almost entirely proof-based. As a result, most equations we derive will have a complete, annotated proof preceding them. I have also included what I think are relatively difficult problems, hopefully providing "TJ-level" test questions that driven, yet usually nervous, students so fervently desire. Although there are no guarantees, I believe that if one does all the examples and problems and **truly understands** them then they will be in good shape for the relevant test. *Emphasis on the 'truly understand' part.*

Now, the other issue I hope to address: I feel there has been some deficiency and inherent unfairness in the TJ system for some time; for the most part, only students who are willing to sacrifice large portions of their summers and come from families financially well off enough to pay for summer school can learn and get ahead. The rest of the students aren't "left behind" but, based on my observation and experience, are certainly at a disadvantage compared to their peers. This has always bothered me and I'd like to do a small part to address this situation constructively and level the playing field.

This book is structured in the same order as the material one would learn throughout TJ Math 4/5 as taught over the last couple years. If you are a non-TJ student using this book and don't know where a certain topic is (or feel that I have omitted it completely), feel free to contact me through www.aidanwiteck.com. Problems with a star (★) are problems that a TJ math test will likely **not** expect you to be able to do. However, you will have the skills to do them (unless otherwise noted), and they are simply included for your enjoyment.

Lastly, I would like to address a question that I anticipate a lot of this book's users asking: "Where is the solution manual?" The short answer is that there isn't one...yet. By the time this book is out and public, I will be working on a solution manual. On balance, I decided that it is just better to get this out so students can use it, rather than waiting longer to put both this book and the solution manual out at the same time. That being said, if you have a question on a problem (or problems), feel free to contact me and I will do my best to send promptly a picture of a handwritten solution to it.

Best of luck in the upcoming year and I hope you find this to be a valuable resource.

Aidan Witeck

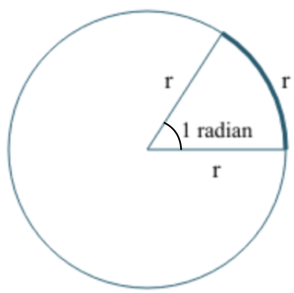
Contents

1 Trigonometry Intro

This chapter will go over some of the functions that you will use throughout your math career. We will first start by reviewing the three main trigonometry functions: sine, cosine, and tangent, which we abbreviate as $\sin\theta$, $\cos\theta$, and $\tan\theta$, respectively. Then, we will move to the inverse of each of those functions, cosecant, secant, and cotangent. We denote these as $\csc\theta$, $\sec\theta$, and $\cot\theta$, respectively.

1.1 Radians

As it turns out, there is more than one way to describe an angle. The way you've probably used your whole life is degrees. With all due respect to humans, that is one of the worst ways to describe an angle. Who picked 360 degrees to be a full rotation? There is nothing mathematically natural about this number. Sure, it is divisible by a lot of numbers, but it is not a natural way to quantify an angle. Through the majority of this course, you will be using a new way of describing your angles: the radian! What exactly is a radian? A radian is the angle that produces an arc length of one radius. This is roughly 57.296°.



The Radian

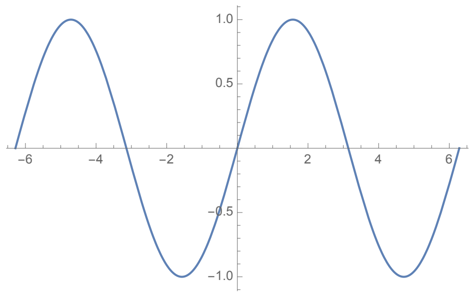
In addition, we can convert between degrees and radians relatively easily. All we have to do is a little dimensional analysis and multiply our degrees by $\frac{\pi}{180^\circ}$, or if we are converting from radians to degrees, $\frac{180^\circ}{\pi}$. Why does this work? Because π radians is equal to 180°. This makes sense, because our circumference is $2\pi R$. If we want the arc length of 180 degrees, then our arc length is half of the circumference, or πR. By the definition of a radian, this means our angle is π radians.

1.2 The Sine Function

The sine function gives you the ratio of the "opposite" side of the triangle to its hypotenuse. The "opposite" side is the side of the triangle across from the angle you are focusing on (this angle is the θ in $\sin\theta$. In addition, although not necessary in this course $\sin\theta$ can be represented by the infinite series (also known as a Taylor Series)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!}\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\theta^{2n-1}}{(2n-1)!}$$

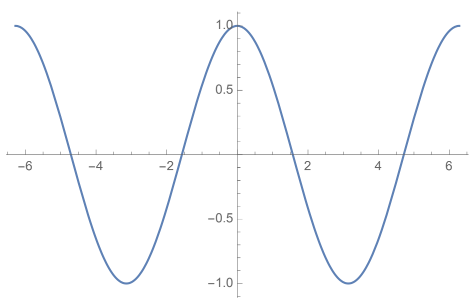
Although there is no easy way to find the sine for any value of theta without using the infinite series above, there are some useful tricks that we will learn later. If you are expected to evaluate the sine of some really ugly number, it will most likely be in the calculator section portion of your test (fun fact: calculators use this series up to a certain number of terms when calculating $\sin\theta$). An important thing to note about the sine function is it's range. The minimum and maximum values of this function are -1 and 1, respectively. Thinking about this in terms of a ratio of the opposite side to the hypotenuse, this makes sense. A leg of a right triangle will always be smaller than its hypotenuse. Because this, the ratio can never exceed 1 (or go below -1). The sine function function oscillates, meaning that when graphed, it looks like a wave (see below). This will become much more clear when we go over how to graph trigonometric functions.



$f(\theta) = \sin\theta$

1.3 The Cosine Function

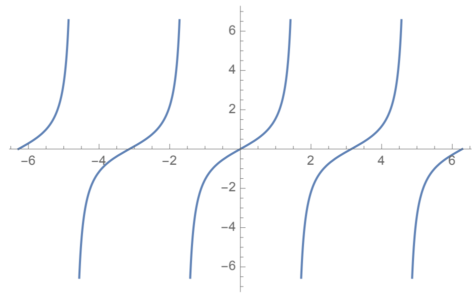
The cosine operator gives you the ratio of the adjacent side of the triangle to its hypotenuse. Similar to the sine function, it is a ratio between -1 and 1. There's nothing much different about the cosine function, except that its maximum and minimum values occur at different values of θ when compared to the sine function. Again, this will become much more clear when we go over graphing.



$f(\theta) = \cos\theta$

1.4 The Tangent Function

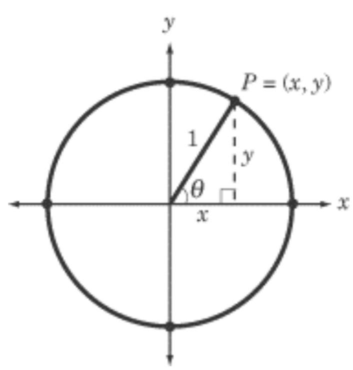
Tangent is not what it seems. It has nothing to do with tangent line. Rather, it is a ratio between the opposite side and the adjacent side of a right triangle. Something important to note about the tangent function is that its range is different than the sine and cosine functions; it covers all y-values, from $-\infty$ to ∞. This is because, unlike sine and cosine functions, where the length of the hypotenuse is always greater than the opposite or adjacent side, the hypotenuse is not involved in the tangent function. You could have a triangle with an extremely small adjacent side and a big opposite side, and $|\tan\theta|$ would be huge. If the opposite side was small and the adjacent large, then $|\tan\theta|$ would be very small.



$f(\theta) = \tan\theta$

1.5 The Unit Circle

The unit circle is a circle with radius 1. Although this may not seem special, the unit circle allows to easily find trig values. Logically, the next question would be "Ok, so how does a circle relate to a right triangle?" As it turns out, if you set up a right triangle in a circle a certain way, with the hypotenuse of the triangle equal to the radius of the circle, your $\cos\theta$ value is simply the horizontal, or x, component of the triangle. Look at the following image:



For any given value of θ, $\cos\theta$ will be the value of the adjacent side, x, over the hypotenuse, which is 1. Therefore, $\cos\theta$ is the x value. The same logic applies to $\sin\theta$ as well, except it is the y value, because it is the opposite side, y, over the hypotenuse, which is 1. This allows us to find the values $\sin\theta$, $\cos\theta$, and as a result, $\tan\theta$... by using a right triangle and circle! Using the properties of 30-60-90 right triangles, we can prove that $\sin 30° = \frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, and $\cos 60° = \frac{1}{2}$. In addition, we can find the trig values for 45°. In addition, something very important to note is the significance of the tangent function in the unit circle. Tangent is the ratio of the opposite side (y) to the adjacent side (x). We can substitute in x and y, which gives us

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1}$$

This equation can be generalized to any right triangle, because if the hypotenuse is not equal to one, they still cancel out ($\frac{\frac{\text{Opposite}}{\text{Hypotenuse}}}{\frac{\text{Adjacent}}{\text{Hypotenuse}}} = \frac{\text{Opposite}}{\text{Adjacent}}$). Lastly, another important thing to realize about $\tan\theta$ is that it gives you the slope of the hypotenuse of the triangle in the unit circle. This means that if you had a hypotenuse with a slope of 2, then $\tan\theta = 2$.

1.6 Cosecant, Secant, and Cotangent

Cosecant is just the reciprocal of the sine function. Namely,

$$\csc\theta = (\sin\theta)^{-1} = \frac{1}{\sin\theta} \tag{2}$$

This is not to be confused with the inverse sine function, $\sin^{-1}\theta$**.** The cosecant function gives you the ratio of the hypotenuse to the opposite side. As you may have assumed, the range of this function is different from the sine function as well. In fact, its almost the opposite (both functions come into contact with the line $y = 1$)! The range of $f(x) = \csc\theta$ is $(-\infty, -1] \cup [1, \infty)$. Similarly, the secant function is

$$\sec\theta = (\cos\theta)^{-1} = \frac{1}{\cos\theta} \tag{3}$$

which is the ratio of the hypotenuse to the adjacent side. The range of this is the same as the range of $f(x) = \csc\theta$. However, it has different values of θ that give the function its extrema. Lastly, the cotangent function is the reciprocal of the tangent function.

$$\cot\theta = (\tan\theta)^{-1} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{4}$$

This is the ratio of the adjacent side to the opposite side.

1.7 Inverse Trigonometric Functions

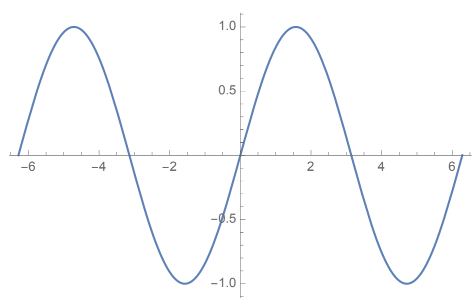
In addition to the normal trig functions that take a measure of an angle and give a ratio, there are also inverses of those functions - namely, they take a ratio and give an angle. We denote these functions (take the inverse sine function for example) as $\sin^{-1}\theta$ or $\arcsin\theta$. Consider the example $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$. We are looking for an angle whose sine value is $\frac{\sqrt{2}}{2}$. Hopefully, you realized that angle is $\frac{\pi}{4}$... or $\frac{3\pi}{4}$, $\frac{-5\pi}{4}$, $\frac{9\pi}{4}$, the list goes on. Because of this, the range of the inverse trig functions must be restricted. If we didn't have a restricted domain, the inverse trig functions would not be functions (they wouldn't pass the vertical line test) and would be completely useless! However, we have fixed this by restricting the range of these functions. Below are the inverse trigonometric functions and their ranges.

Table 1: Ranges of Inverse Trig Functions

Function	**Range**
$y = \sin^{-1}\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$y = \cos^{-1}\theta$	$0 \leq \theta \leq \pi$
$y = \tan^{-1}\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$y = \csc^{-1}\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, y \neq 0$
$y = \sec^{-1}\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, y \neq \frac{\pi}{2}$
$y = \cot^{-1}\theta$	$0 \leq \theta \leq \pi$

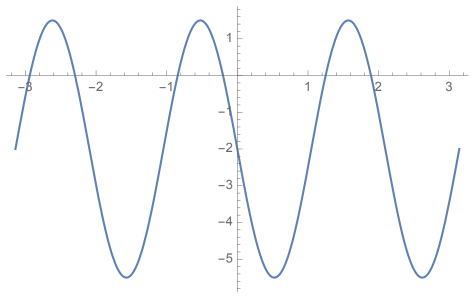
1.8 Graphs of Trigonometric Functions

Probably the most difficult part of this unit is graphing trig functions. This is something that takes a lot of practice to fully master - but, once you have done that, you'll be well off for the test. The generic graphs of the six trigonometry functions are straightforward, but as you apply transformations, graphing becomes tedious very quickly. Let's begin with the graph of $y = \sin x$:

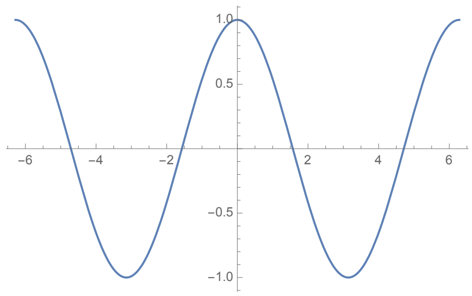


The zeros of this graph are -2π, $-\pi$, 0, π, and 2π. More generally, because this graph oscillates in this way over an infinite interval, the x intercepts are $k\pi$, where $k \in \mathbb{Z}$

The more general form for a sine function is $f(x) = A\sin[B(x-C)] + D$, where A is the amplitude, B is the frequency, C is the horizontal (or phase) shift, and D is the vertical shift. As you may imagine, graphing a function with all of these transformations can be challenging. Consider the function $f(x) = \frac{7}{2}\sin[3(x-\frac{\pi}{3})] - 2$. Unless you're a pure genius, you can't look at this function and graph it immediately. Rather, you have to analyze the function, write down all of its features, plot critical points, and then graph it. Once you do that, the wave, if done correctly, should look something like this:

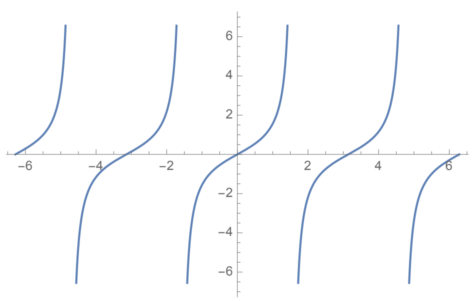


The cosine graph is very similar to the sine graph. In fact, it is the exact same, just shifted horizontally. The graph for $f(x) = \cos x$ is the following:



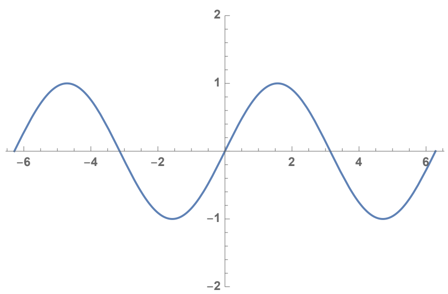
Although you may have figured it out by now, the zeros for this graph are $\frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$. Again, the cosine function has the same general form as the sine function. $f(x) = A\cos[B(x-C)] + D$, where each of the constants mean the same thing as above.

The tangent function has a graph shape quite different compared to the graphs of $\sin\theta$ and $\cos\theta$. The graph of $f(x) = \tan x$ looks like:

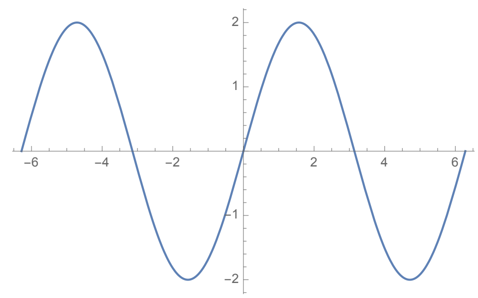


The asymptotes on this graph are $\frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$. This makes sense, too, because $\tan x = \frac{\sin x}{\cos x}$, and when $x = \frac{\pi}{2} + k\pi$, $\cos x = 0$, which makes our denominator zero, resulting in $\tan\theta$ growing infinitely. Another feature of this graph is that it's period is π, where in the sine and cosine graphs, our period is 2π. The period is of a function is how long it takes to make one complete cycle (in our case, we are concerned with the x distance it covers in one complete cycle). Although the graph of $f(x) = \tan x$ differs from the other two, it has the same general form as the others: $f(x) = A\tan[B(x-C)] + D$.

Before we go over the cosecant, secant, and cotangent graphs, it is important to know that there are several transformations we can do to our three main trig functions. The first constant, A, is known as the amplitude of the function. This is simply a vertical dilation, which is pretty easy to visualize. Consider the graphs of $f(x) = \sin x$ and $f(x) = 2\sin x$.



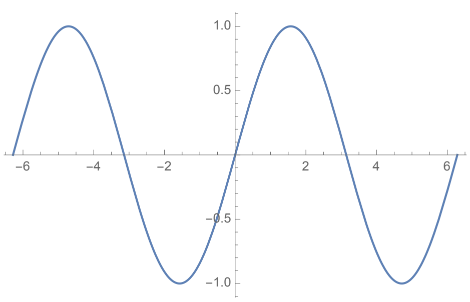
$f(x) = \sin x$

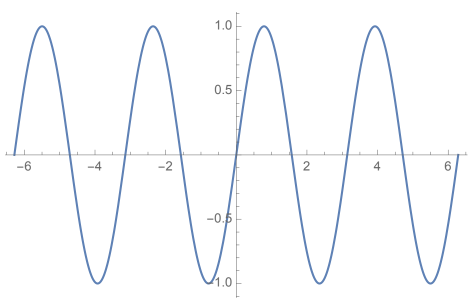


$f(x) = 2\sin x$

As you can see, the graph of $f(x) = 2\sin x$ has twice the amplitude as $f(x) = \sin x$. Its range is $[-2,2]$ compared to the range of $f(x) = \sin x$, which is $[-1,1]$.

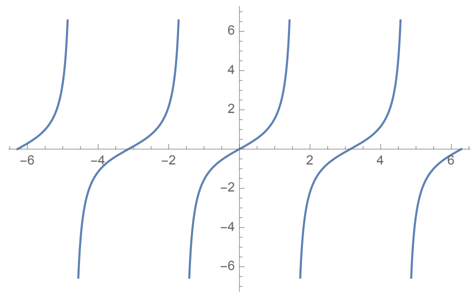
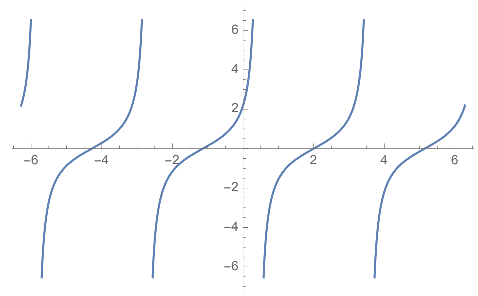
Another transformation that cam be applied is the frequency, f. Frequency is defined as $\frac{1}{period}$. Since the period is $\frac{2\pi}{f}$, B is $2\pi f$, or $\frac{2\pi}{period}$. This transformation that we get out of this is a horizontal dilation. B is inversely related, however, with period. As B increases, period decreases and we horizontally compress it. As B decreases, period increases and we horizontally stretch it. To show this graphically, compare the graphs of $f(x) = \sin x$ and $f(x) = \sin(2x)$





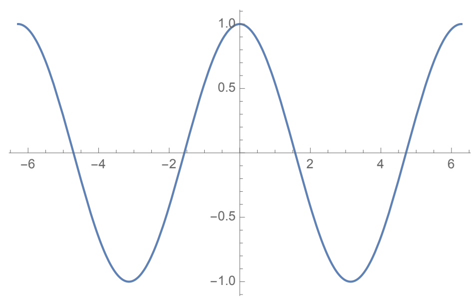
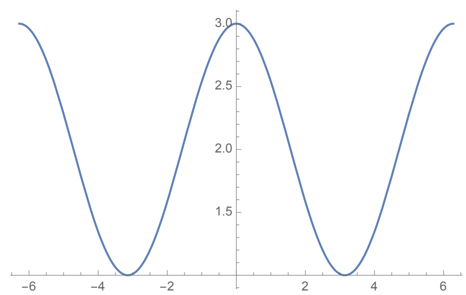
$B = 2$, which means that $2 = \frac{2\pi}{period}$. Therefore, our period is π, which the graph agrees with.

We can also horizontally shift a function. The value that tells us our horizontal shift is C, which is how much **to the right** we move our graph. It is important to note in our general sine equation that $A\sin[B(x-C)]+D$, meaning that if we have $x+C$, the graph will shift to the left. Consider the graphs $f(x) = \tan x$ and $f(x) = \tan(x-2)$

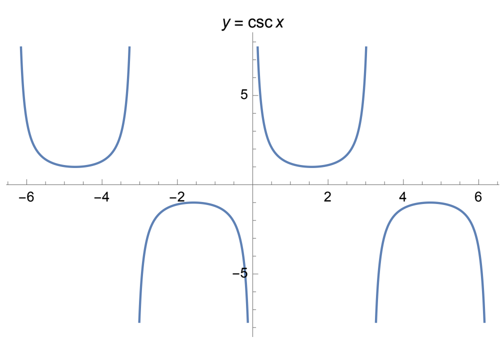
Our C value is 2, which means that the graph shifts two units to the right.

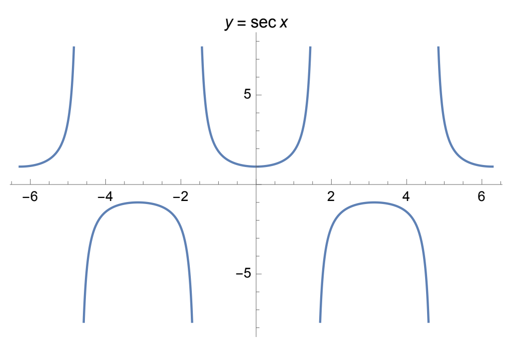
Finally, the last transformation we will cover is the vertical shift, or D value. This is pretty straightforward. D is the number of units up that you shift the whole graph. Consider the graphs $f(x) = \cos x$ and $f(x) = \cos x + 2$

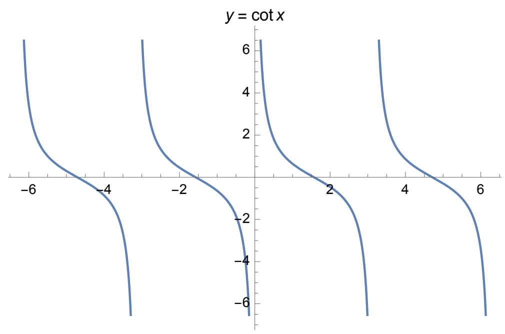



D=2, which means that we shift the graph up two units.

Now we will go over the graphs of $f(x) = \csc x$, $f(x) = \sec x$, and $f(x) = \cot x$. The graphs of these functions are:



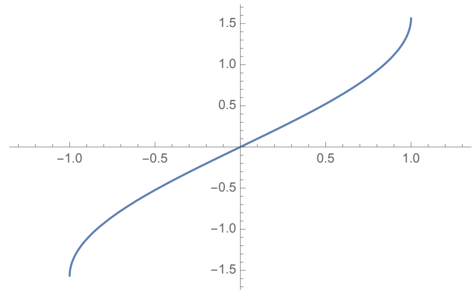




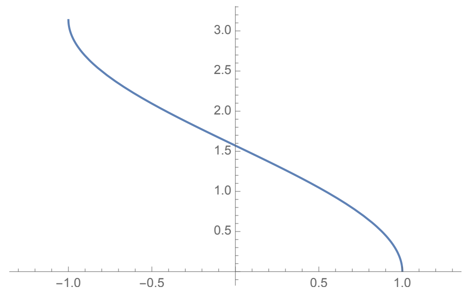
In the graph $y = \csc x$, our vertical asymptotes are the lines $x = k\pi$, where $k \in \mathbb{Z}$. In the graph $y = \sec x$, our vertical asymptotes are the lines $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$. Finally, in the graph of $y = \cot x$, our vertical asymptotes are the lines $x = k\pi$ where $x \in \mathbb{Z}$. There is an interesting pattern to notice here: the x-coordinates of our vertical asymptotes are the x-coordinates of the zeros its inverse has. For example, because $f(x) = \csc x$ is the inverse of $f(x) = \sin x$, it has vertical asymptotes at the zeroes of $f(x) = \sin x$, which are $x = k\pi$, where $k \in \mathbb{Z}$. One thing to note, however, is that our period stays the same. The period of $\csc x$ is the same as the period for $\sin x$, and similarly for $\sec x$ and $\cos x$. Although the cotangent graph may be a little confusing at first, all you do is reflect $\tan x$ over the y-axis and shift $\frac{\pi}{2}$ to the right. Namely,

$$\cot x = \tan\left(-x - \frac{\pi}{2}\right) \tag{5}$$

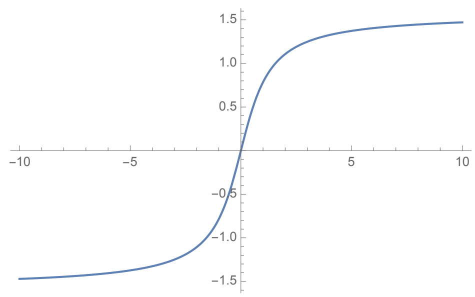
Although you most likely won't be tested on this, it is worth going over the graph of the inverse trigonometry functions. In essence, they're their parent graphs rotated 90 degrees (or $\frac{\pi}{2}$ radians!!) and with a restricted domain. Let's take a look at the graph of $y = \sin^{-1} x$:



Our domain is -1 to 1, which makes sense (that was the range of $\sin x$). Recall from Algebra II how the domain of an inverse, $f^{-1}(x)$ is the range of its function $f(x)$. In addition our range is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, which also makes sense. In addition, if we pick any x-coordinate in the graph (let's go with 1/2), then our corresponding y-value will be the angle who's sine value is 1/2. Looking at the graph, we see the y-value is about 0.5. Our actual y-value is approximately 0.524, or $\frac{\pi}{6}$, so that checks out as well. Now, let's look at $\cos^{-1} x$:



This is similar to the graph of $\sin^{-1} x$, except we apply two transformations. First, we reflect $\sin^{-1} x$ over the y-axis, and then shift it up $\frac{\pi}{2}$ units. Therefore, we can say that $\cos^{-1} x = \sin^{-1}(-x) + \frac{\pi}{2}$. Feel free to test this yourself if this doesn't make sense. Lastly, we will look at the graph of $\tan^{-1} x$:

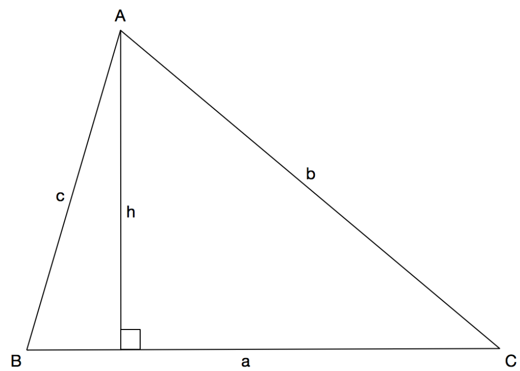


We have horizontal asymptotes of $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$. Again, this is because we have a restricted range. This graph looks qualitatively different, but its purpose is still the same: our x-value returns a y-value, which is the angle who's tangent gives us our x-value. [1]

1.9 Law of Sines

Sometimes when we're finding angles and sides of a triangle, we won't be dealing with right triangles. Because of this, we can't simply plug in (hypotenuse $* \cos\theta$ to find the length of the adjacent side, because that works only if the triangle has a 90 degree angle. So, we have to improvise. Lucky for us, there is a formula we can use to quantify a non-right triangle. Consider an acute triangle $\triangle ABC$, where h is an altitude:

[1] Although there are graphs of $\csc^{-1} x, \sec^{-1} x$, and $\cot^{-1} x$, you will most likely never hear about these in a Pre-Calculus course.



Note that a covers the whole bottom side of the triangle.
Starting off, we know that *A, B,* and *C* are angles. Therefore,

$$\sin B = \frac{h}{c} \;;\; \sin C = \frac{h}{b}$$
$$h = c\sin B \;;\; h = b\sin C$$
$$c\sin B = b\sin C$$
$$\therefore \frac{c}{\sin C} = \frac{b}{\sin B}$$

The same process can be repeated with drawing an altitude from B, and we will find that

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

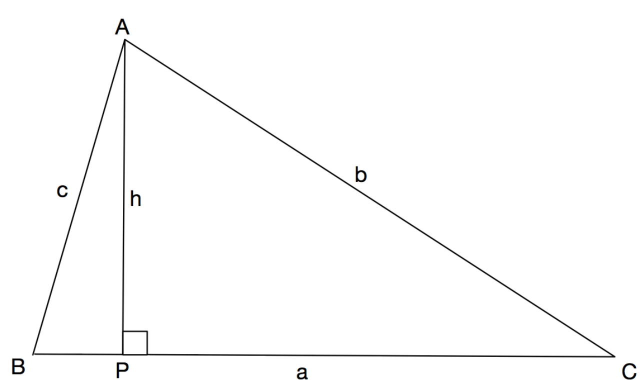
Then, we have proven the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{6}$$

This law holds true for any triangle. It is important to note that there is a ratio between the sides of **any triangle** and the sine of their corresponding angle.

1.10 Law of Cosines

Now, to prove the law of cosines, consider the figure below:



Starting off, we know that $\cos B = \frac{BP}{c} \Rightarrow$ c$(\cos B) = BP$. Thus,

$$a = BP + PC$$

$$PC = a - c(\cos B)$$

$$\sin B = \frac{AP}{c} \Rightarrow AP = c \sin B$$

Because of the Pythagorean theorem,

$$b^2 = PC^2 + AP^2$$

$$b^2 = [a - c(\cos B)]^2 + (c \sin B)^2 = a^2 + c^2 \cos^2 B - 2ac(\cos B) + c^2 \sin^2 B$$

Rearranging,

$$b^2 = c^2 \cos^2 B + c^2 \sin^2 B - 2ac(\cos B) + a^2$$

$$b^2 = c^2(\cos^2 B + \sin^2 B) - 2ac(\cos B) + a^2$$

Because of Pythagorean identities (which we will study in the next chapter), $\cos^2 B + \sin^2 B = 1$

$$b^2 = c^2 - 2ac(\cos B) + a^2$$

We have now proved the law of cosines. It turns out that this works for all sides (given that we use different altitudes when proving the law). Thus, given any sides of a triangle a, b, and c and corresponding respective angles A, B and C,

$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

In addition, if we are given all the sides of an angle, we can use the Law of Cosines to find the angles. Given a, b, and c, we can rearrange the Law of Cosines to get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

Just like the Law of Cosines, this rule applies to any angle of a triangle.

1.11 Problems

1. Find the value of $\tan\frac{\pi}{6} - \sin\left[\arccos\left(-\frac{1}{2}\right)\right]$

2. A block is attached to a spring, which is attached to the floor. The spring makes the block go up and down. When the block is at its maximum height, the spring is 7cm in length. When the block is at its minimum height (i.e. the spring is at its most compressed point), the block is 2cm off the ground. Assume the block's height keeps oscillating in this way for an infinite amount of time. If the block starts at its maximum height and is then released, it takes 3.0s for it to reach its maximum height again, and then go down to its minimum height.

   (a) Find an equation for the block's height as a function of time.

   (b) How high off the ground is the block after t=5.0s?

3. One of the things our Mars Rover does is give us statistics on the climate of Mars. Throughout the year 2017, the maximum temperature on Mars, recorded on February 12th, was 68°F. The minimum temperature, recorded on August 12th, was -81°F. Find a cosine function that represents the temperature, in degrees Fahrenheit, of Mars as a function of time in months.

4. Graph the following equations:

   (a) $f(x) = 3\sin\left(6x - \frac{\pi}{4}\right) + 2$

   (b) $f(x) = -2\sec\left[3\left(x - \frac{\pi}{2}\right)\right] - 4$

   (c) $f(x) = \sin x + \cos x$

   (d) $f(x) = 3\tan(6x - \frac{\pi}{4}) + 5$

   (e) $f(x) = -5\cos(8x - \frac{3\pi}{4}) + 1$

   (f) $f(x) = -3\cot(2x - \frac{\pi}{2}) - 2$

5. Find the values of the six trig functions (namely, $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$) of an angle, θ, in standard position that goes through the point $(-4, 5)$. Rationalize your answers.

6. A 10 foot stop sign is located 110 feet from an observer. A light pole is placed another 36 feet beyond the stop sign. What is the minimum height the light pole must be in order for the observer to see it over the stop sign?

7. True or False:

   (a) The asymptotes of the graph $y = 2\csc(2x) - 1$ are the x-intercepts of the graph of $y = 2\sin(2x) - 1$

   (b) The local maxima of any transformed secant graph will always be $\frac{\pi}{2}$ to the right of the local minima of its corresponding transformed sine graph.

   (c) $\tan x = -\cot(x + \frac{\pi}{2})$

(d) $\sin^{-1}(\sin \frac{2\pi}{3})$ is undefined.

8. Rewrite the following function in terms of x and without trigonometric functions:

$$f(x) = \csc\left(\cos^{-1}\frac{3}{x}\right)$$

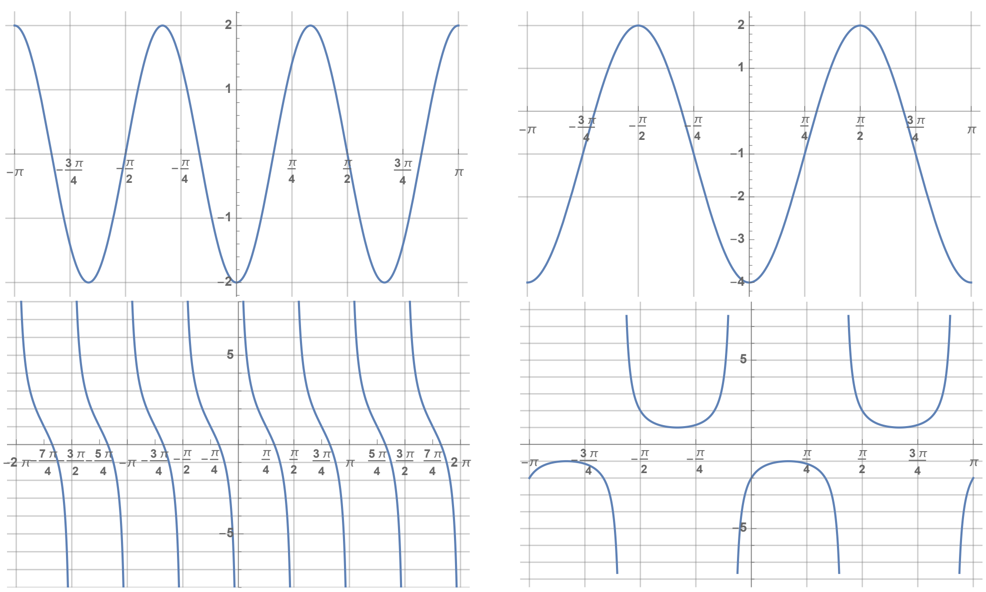
9. Given a triangle with side lengths $a = 3, b = 9, c = 7$, find the angle measures A, B, and C.

10. Given a triangle with side lengths $a = 4, b = 7$, and angle measure $C = 73°$, find all remaining sides and angles.

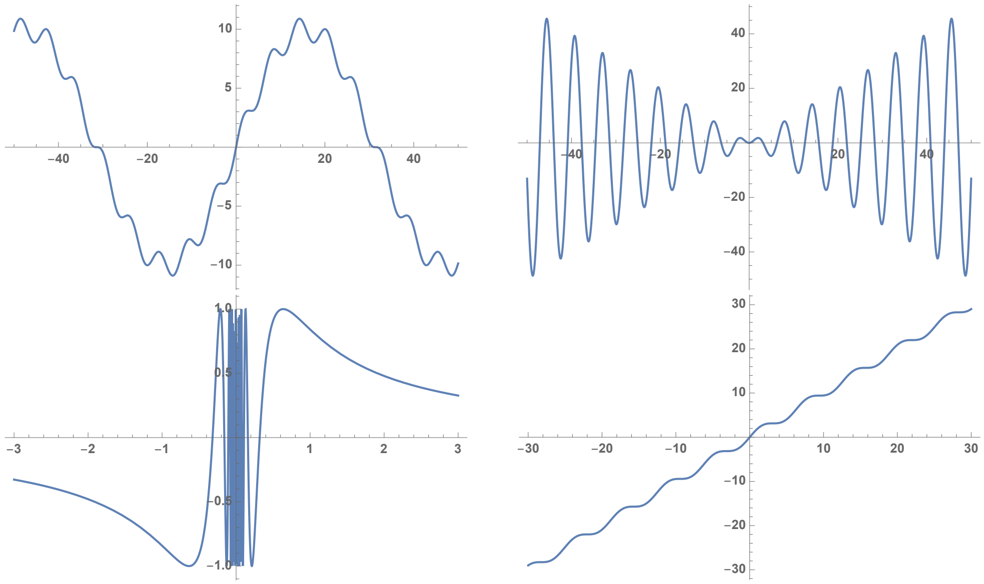
11. What is the period of the function $f(x) = \tan x - \cot x$?

12. Starting at the origin, a beam of light hits a mirror (in the form of a line) at point $A = (4,8)$ and is reflected to point $B = (8,12)$. Compute the exact slope of the mirror.

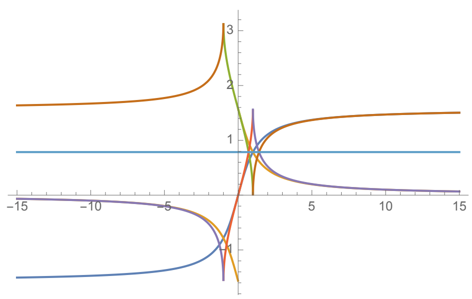
13. For each graph shown below, find a) the amplitude, b) the period, c) the frequency, d) any shifts, and e) the equation.



14. ★ Come up with a general equation for the following graphs - coefficients are not the most important thing here.



15. ★ Pictured below is a graph of several functions: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\csc^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$. However, as you may notice, there is a symmetry to these graphs: if all of these were one graph, it would be symmetrical about the line $y = \frac{\pi}{4}$, which is the light blue line. Your job is to prove why this happens, using any algebraic method you may know.



2 Trigonometric Identities

Now that we have a decent understanding of what each trigonometric function means, there are several trigonometric identities that are worth learning. This is probably the most algebra-intensive unit of Pre-Calculus. Trig identities help us in numerous ways, one of them being their ability to simplify problems that contain bulky equations with trig functions in them.

2.1 Pythagorean Identities

One of the main identities of trigonometry functions are the Pythagorean identities. Recall that when there is a right triangle in the unit circle, the x-component is $\cos\theta$ and the y-component is $\sin\theta$. In addition, because of the Pythagorean theorem, $x^2+y^2=1$ (because the hypotenuse in the unit circle is 1). Substituting for x and y, we get our main Pythagorean identity:

$$\sin^2\theta+\cos^2\theta=1$$

With simple algebra, two other identities can be derived:

$$\sin^2\theta+\cos^2\theta=1$$
$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$
$$1+\cot^2\theta=\csc^2\theta$$

$$\sin^2\theta+\cos^2\theta=1$$
$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\tan^2\theta+1=\sec^2\theta$$

In summary, the three Pythagorean Trig identities are:

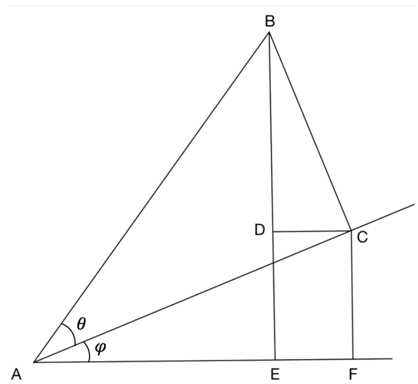
$$\sin^2\theta+\cos^2\theta=1 \tag{7}$$

$$1+\cot^2\theta=\csc^2\theta \tag{8}$$

$$\tan^2\theta+1=\sec^2\theta \tag{9}$$

2.2 Addition and Subtraction Identities

In addition to Pythagorean Identities, there are Double-Angle identities, which make it easier to solve a problem that involves trig functions such as $\sin(\theta+\phi)$. As it turns out, there are formulas that can be used to simplify this. The prove this, consider the figure below:



Given $AB = 1, BC \perp AC, BE \perp AF, CF \perp AF$, and $DC \| EF$, we know that ACB, DCF, CDE, DEF, and CFE are right angles. Thus, $ACF = \frac{\pi}{2} - \phi$, which means that $ACD = \phi$. Since ACB is a right angle, $BCD = \frac{\pi}{2} - \phi$, which means that $DBC = \phi$.

$$BC = AB \sin\theta$$

$$AC = AB \cos\theta$$

Since *DCFE* is a rectangle, *DE*=*CF* and *DC*=*EF*

$$\sin\phi = \frac{CF}{AC} \Rightarrow CF = AC\sin\phi = AB\sin\phi\cos\theta$$

$$\cos\phi = \frac{BD}{BC} \Rightarrow BD = BC\cos\phi = AB\cos\phi\sin\theta$$

Considering $\triangle ABE$, we can deduce that

$$\sin(\theta+\phi) = BE = BD + DE = AB\sin\phi\cos\theta + AB\cos\phi\sin\theta$$

However, since $AB = 1$, we can take that out of the equation to get our sum of angle sine identity:

$$\sin(\theta+\phi) = \sin\phi\cos\theta + \cos\phi\sin\theta \tag{10}$$

Now, to prove what $\cos(\theta+\phi)$ is:

$$BC = AB\sin\theta$$

$$AC = AB\cos\theta$$

$$\cos\phi = \frac{AF}{AC} \Rightarrow AF = AC\cos\phi = \cos\theta\cos\phi$$

$$\sin\phi = \frac{DC}{BC} \Rightarrow DC = BC\sin\phi = \sin\theta\sin\phi$$

Using the same logic as above, we can deduce that

$$\cos(\theta+\phi) = AE = AF - EF = AB\cos\theta\cos\phi - AB\sin\theta\sin\phi$$

And since $AB = 1$, we can take that out, giving us our sum of angle cosine identity:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{11}$$

Now that we have these two, the tangent sum of angles identity is easy:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\phi\cos\theta + \cos\phi\sin\theta}{\cos\theta\cos\phi - \sin\theta\sin\phi} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \tag{12}$$

Now that we have proven the addition identities, they can be used to prove subtraction identities. However, we need to know the following in order to do this:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

If this doesn't make sense, think about each of the functions graphically. Since the cosine function is symmetrical about the y-axis (a.k.a. an even function), taking an x-value of $-\theta$ will yield the same y-value as θ. On the other hand, the sine function is not symmetrical about the y-axis . However, it is symmetrical about the origin (we call this an odd function, more on this in Chapter 6). This means that if we compare the y-values of $-\theta$ and θ, they will be opposite of each other. The tangent function can be verified by the sine and cosine functions:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Now that that's out of the way, $\sin(\theta - \phi)$ can be proven relatively easily:

$$\begin{aligned}\sin(\theta - \phi) &= \sin(\theta + (-\phi)) \\ &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi) \\ &= \sin\theta\cos\phi - \cos\theta\sin\phi\end{aligned}$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \tag{13}$$

The cosine subtraction identity can be proved similarly:

$$\begin{aligned}\cos(\theta - \phi) &= \cos(\theta + (-\phi)) \\ &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi) \\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{14}$$

The tangent function, like before, is proven by sine and cosine:

$$\begin{aligned}\tan(\theta-\phi) &= \frac{\sin(\theta-\phi)}{\cos(\theta-\phi)} \\ &= \frac{\sin\theta\cos\phi-\cos\theta\sin\phi}{\cos\theta\cos\phi+\sin\theta\sin\phi} \\ &= \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}\end{aligned}$$

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} \tag{15}$$

2.3 Double-Angle Identities

Because the angle sum and subtraction identities have been proven, the double-angle identities become easy to prove. To start off with $\sin(2\theta)$:

$$\begin{aligned}\sin(2\theta) = \sin(\theta+\theta) &= \sin\theta\cos\theta+\sin\theta\cos\theta \\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{16}$$

The cosine double angle identity can be proven with the same method as well:

$$\begin{aligned}\cos(2\theta) = \cos(\theta+\theta) &= \cos\theta\cos\theta-\sin\theta\sin\theta \\ &= \cos^2\theta-\sin^2\theta\end{aligned}$$

$$\cos(2\theta) = \cos^2\theta-\sin^2\theta \tag{17}$$

Finally, the tangent function:

$$\begin{aligned}\tan(2\theta) &= \tan(\theta+\theta) \\ &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} \\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{18}$$

2.4 Co-function Identities

Co-function identities relate two trig functions together based on their arguments (an argument is the angle, usually θ. For $\cos(\frac{\pi}{2}-\theta)$, the argument is $\frac{\pi}{2}-\theta$). Compared to other identities, these should be relatively easy to understand because we can prove them with either angle-sum identities or by graphing. One co-function identity is:

$$\sin\theta = \cos(\frac{\pi}{2}-\theta)$$

Thinking about this visually, $\cos(\frac{\pi}{2}-\theta)$ applies a horizontal shift of $\frac{\pi}{2}$ units to the right to the cosine graph. When this occurs, we have a sine graph. If that doesn't make sense, it can be proven by angle subtraction identities:

$$\begin{aligned}\cos(\frac{\pi}{2}-\theta) &= \cos\frac{\pi}{2}\cos(-\theta) - \sin\frac{\pi}{2}\sin(-\theta)\\ &= -\sin\frac{\pi}{2}\sin(-\theta)\\ &= \sin\theta\end{aligned}$$

$$\cos(\frac{\pi}{2}-\theta) = \sin\theta \tag{19}$$

There are several co-function identities, but they can all be solved by this method. The other five are:

$$\sin(\frac{\pi}{2}-\theta) = \cos\theta \tag{20}$$

$$\tan(\frac{\pi}{2}-\theta) = \cot\theta \tag{21}$$

$$\cot(\frac{\pi}{2}-\theta) = \tan\theta \tag{22}$$

$$\sec(\frac{\pi}{2}-\theta) = \csc\theta \tag{23}$$

$$\csc(\frac{\pi}{2}-\theta) = \sec\theta \tag{24}$$

2.5 Half-Angle Identities

The half-angle identities may appear daunting at first (mainly because they involve a square root), but they are pretty easy to understand if they're proven with double-angle identities. Starting off with $\cos(\frac{\theta}{2})$:

$$\cos\theta = \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

Using Pythagorean identities,

$$\begin{aligned}
&= \cos^2\frac{\theta}{2} - (1-\cos^2\frac{\theta}{2}) \\
&= 2\cos^2\frac{\theta}{2} - 1 \\
\Rightarrow \cos\theta &= 2\cos^2\frac{\theta}{2} - 1 \\
\frac{\cos\theta + 1}{2} &= \cos^2\frac{\theta}{2}
\end{aligned}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta+1}{2}} \tag{25}$$

Now, for $\sin\frac{\theta}{2}$:

$$\begin{aligned}
\cos\theta = \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\
&= \left(1-\sin^2\frac{\theta}{2}\right) - \sin^2\frac{\theta}{2} \\
&= 1 - 2\sin^2\frac{\theta}{2} \\
\Rightarrow \cos\theta &= 1 - 2\sin^2\frac{\theta}{2} \\
\sin^2\frac{\theta}{2} &= \frac{1-\cos\theta}{2}
\end{aligned}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{26}$$

It is important to note that the $\pm$ symbol does not indicate two solutions, but that the solution could be either positive or negative **depending on where the angle is in the unit circle**. For example, if we are solving for $\cos\frac{\pi}{8}$, we can use our formula to get $\cos\frac{\pi}{8} = \pm\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}$. However, since $\cos\frac{\pi}{8}$ will definitely be positive (think about it in the unit circle!), we know that the answer will be $+\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}$

Finally, for $\tan\frac{\theta}{2}$:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{\cos\theta+1}{2}}}$$

$$= \pm\sqrt{\frac{1-\cos\theta}{\cos\theta+1}}$$

Again, the same rule regarding the $\pm$ sign applies to $\tan\frac{\theta}{2}$.

Summarizing,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta+1}{2}} \qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{\cos\theta+1}}$$

2.6 Problems

Note: This test usually has few application problems on it (it really shouldn't be a whole unit, honestly). Most of the time, you will be either a) verifying trigonometric identities or b) solving for some value of θ given an equation. As a result, this problem set mainly contains these two types of problems.

1. Verify that the following identities are true:
   (a) $\sec x - \tan x \sin x = \cos x$
   (b) $\tan^2 x \sin^2 x = \tan^2 x - \sin^2 x$
   (c) $\frac{1-\cos x}{\sin x} = \csc x + \cot x$
   (d) $\frac{\tan x + \cot x}{\cot x} = (1-\sin^2 x)^{-1}$
   (e) $\sin(\theta + \frac{3\pi}{2}) = -\cos\theta$
   (f) $\cot(\theta) - \cot(\phi) = \frac{\sin(\phi-\theta)}{\sin\theta \sin\phi}$
   (g) $\cot(4\theta) = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$
   (h) $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$
   (i) $\tan\theta + \cot\theta = \sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta$
   (j) $\sin\alpha + \sin\beta + \sin\gamma = 4\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\cos(\frac{\gamma}{2})$
   (k) $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 4\sin\alpha\sin\beta\sin\gamma$
   (l) $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) = \sec\theta + \csc\theta$
2. Write $\cos(\cos^{-1} 4x - \tan^{-1}\sqrt{3})$ as an algebraic expression involving no trigonometric functions.
3. Solve the following equations:
   (a) $4\sec(3\theta + \frac{11\pi}{6}) = -4\sqrt{2}$
   (b) $\sin x + 5\cos x + 6\cos^2 x + 1 = 0$
   (c) $\sin x - \cos x = 4(\sqrt{3}-1)\cos^3 x$
   (d) $2\cos^2 x - \cos x - 1 = 0$
   (e) $\sec^2 x - 2\tan x = 4$
   (f) $2\cot^2 x + \csc^2 x - 2 = 0$
   (g) $\dfrac{\sin x - \cos x}{\sin x + \cos x} = 1 - \sin(2x)$
4. ★ Simplify the following expression:

$$\arcsin(a\sqrt{1-b^2} + b\sqrt{1-a^2})$$

5. ★ Find $\arctan 1 + \arctan \frac{1}{2} + \arctan \frac{1}{3}$ without a calculator.

6. ★ Prove that $\frac{a}{\sin x} + \frac{b}{\cos x} \geq \frac{2\sqrt{ab}}{\sqrt{\sin x \cos x}}$, given that x is an angle such that $0 < x < \frac{\pi}{2}$

3 Matrices

The purpose of this unit is to introduce the basis of linear algebra: matrices and the properties/applications of them. Matrices allow us to neatly arrange a system of linear equations into a "matrix" which has rows and columns of numbers. As we will soon learn, they are extremely useful when solving linear equations. A general rule for solving systems of linear equations is that if we have n equations and n unknowns, then we can solve it. Namely, if we have three unique linear equations [2] and three unknowns (*x, y,* and *z*), then the system can be solved.

3.1 Augmented Matrix Form

Augmented matrices are a combination of two matrices - one matrix for each side of the equation. For example, the following three equations can be put into one matrix and solved, given that they are unique:

$$\begin{array}{r} 2x-3y+10z=2 \\ -5x+4y-z=3 \\ 8x-7y+2z=7 \end{array} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 10 & 2 \\ -5 & 4 & -1 & 3 \\ 8 & -7 & 2 & 7 \end{array}\right]$$

3.2 Row Operations

There are several properties of matrices that will prove to be extremely useful in the sections that follow. Consider the following equations, where *a,b,c,d,e,f,g,h,* and *i* are constants:

$$\begin{array}{c} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{array}$$

As we just learned, this can be represented by the matrix

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right]$$

Going back to the first set of equations, if $ax+by+cz=d$ and $ex+fy+gz=h$, than we can add both equations together (a technique you probably learned in Algebra I). The following equation is:

$$(a+e)x+(b+f)y+(c+g)z=h+d$$

This property can be translated into matrices, which looks something like:

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right] \Rightarrow R_1=r_1+r_2 \Rightarrow \left[\begin{array}{ccc|c} a+e & b+f & c+g & d+h \\ e & f & g & h \\ i & j & k & l \end{array}\right]$$

[2] In the context of this chapter, equations similar to $2x+3y+z=3$ and $4x+6y+2z=6$ are not unique from each other because the second one is just the first equation multiplied by a scalar of 2)

The "$\Rightarrow R_1 = r_1 + r_2 \Rightarrow$" signifies that our new row 1 (capital R) is the sum of the old row 1 and 2 (lowercase r).
In addition to our additive row properties, there is also a multiplicative row property, which is as follows: If $ax+by+cz=d$ than $k(ax+by+cz)=kd$, where k is a constant. In matrix language,

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right] \Rightarrow R_1 = r_1 * k \Rightarrow \left[\begin{array}{ccc|c} ka & kb & kc & kd \\ e & f & g & h \\ i & j & k & l \end{array}\right]$$

Matrix rows can also be switched:

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right] \Rightarrow R_1 = r_2; R_2 = r_1 \Rightarrow \left[\begin{array}{ccc|c} e & f & g & h \\ a & b & c & d \\ i & j & k & l \end{array}\right]$$

3.3 Row echelon form

A matrix is said to be in row echelon form if the diagonal of the matrix is all ones and the numbers below the diagonal are all zero. A matrix in reduced row echelon form if the diagonal of the matrix is all ones and there are no other constants on the left side. Consider the following matrices in row echelon form and reduced row echelon form, respectively:

$$\left[\begin{array}{ccc|c} 1 & -3 & 10 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Clearly, having a matrix in reduced row echelon form makes it much easier to solve; in fact, it is solved! Each variable has a solution. In row echelon form, however, the answer is not as clear. This is where Gauss-Jordan elimination comes into play.

3.4 Gauss-Jordan Elimination

The purpose of Gauss-Jordan Elimination is to use row operations to reduce all of our equations to only contain one variable (reduced row echelon form). For example, consider the augmented matrix A.

$$A = \left[\begin{array}{cc|c} -5 & 4 & 49 \\ 7 & 2 & -23 \end{array}\right]$$

In order to find the solutions to A, we will first need to get it into row echelon form (and then reduced row echelon form).

$$\left[\begin{array}{cc|c} -5 & 4 & 49 \\ 7 & 2 & -23 \end{array}\right] \Rightarrow R_1 = r_1 + r_2 \Rightarrow \left[\begin{array}{cc|c} 2 & 6 & 26 \\ 7 & 2 & -23 \end{array}\right] \Rightarrow R_1 = \frac{1}{2}r_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 13 \\ 7 & 2 & -23 \end{array}\right] \Rightarrow R_2 = r_2 - 7r_1$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 13 \\ 0 & -19 & -114 \end{array}\right] \Rightarrow R_2 = -\frac{1}{19}r_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 13 \\ 0 & 1 & 6 \end{array}\right]$$

Whew, okay. Now we have A in row echelon form. It is clear that y is 6, but now we have to solve for x. We do this by further operating on A to get it into reduced row echelon form:

$$\left[\begin{array}{cc|c} 1 & 3 & 13 \\ 0 & 1 & 6 \end{array}\right] \Rightarrow R_1 = r_1 - 3r_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 6 \end{array}\right]$$

Alas, we have our solution. $x = -5$, and $y = 6$. Although this method may seem like way too much work (given that you can solve it much easier by setting up a system of equations and cancelling out a variable) it proves to be extremely useful when we have three or more variables. For example, try to solve the following system by using Algebra I skills, and then try to solve it by Gauss-Jordan elimination:

$$\begin{gathered} 4x + 3y = 4 \\ 2x + 2y - 2z = 0 \\ 5x + 3y + z = -2 \end{gathered}$$

Hopefully, that example proved the advantage that Gauss-Jordan elimination provides.

3.5 Matrix Multiplication

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad B = \begin{bmatrix} g & h & i \\ j & k & l \\ m & n & o \end{bmatrix}$$

Say we have matrices A and B, and we want to multiply them together. Given the previous row operations learned, it might seem intuitive to think that $A \times B$ is each component multiplied together. However, this is not the case. Instead, each horizontal component in the first matrix (in this case, A), is multiplied by each vertical component in the second matrix (B). Each component product is then summed up to get a final component.

$$AB = \begin{bmatrix} ag + bj + cm & ah + bk + cn & ai + bl + co \\ dg + ej + fm & dh + ek + fn & di + el + fo \end{bmatrix}$$

The top left component of $A \times B$ is the sum of the products of the top row of A and the leftmost column of B. The component to the right is the same row of A but one column to the right in B.

Example 3.1

If consider the following matrices A and B. Find $A \times B$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & -2 & 3 \\ 5 & 2 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (2*-4)+(-1*5) & (2*-2)+(-1*2) & (2*3)+(-1*2) \\ (3*-4)+(-4*5) & (3*-2)+(-4*2) & (3*3)+(-4*2) \end{bmatrix}$$
$$= \begin{bmatrix} -13 & -6 & 4 \\ -22 & -14 & 1 \end{bmatrix}$$

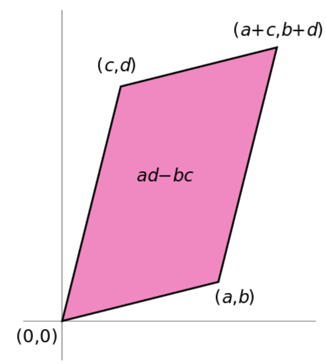
$$AB = \begin{bmatrix} -13 & -6 & 4 \\ -22 & -14 & 1 \end{bmatrix}$$

3.6 Determinants

If we wanted to represent a set of points in a matrix, it would look like the following:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Where each element in the matrix is a component of a point in space (for example, the ordered pair (x_1, y_1) represents a point in a Cartesian plane). The determinant gives us the area of the parallelogram formed by the two points, the origin, and the sum of the two points.



To understand why the area of the parallelogram is $ad - bc$, you will need to have a solid understanding of vectors and projections (we cover that in the next chapter). However, if you already know this, a proof can be found online. We denote a determinant of a matrix by enclosing it in vertical lines, rather than brackets. The determinant matrix A can be written as $|A|$, $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$, or $\det(A)$. It is important to note that the determinant is a scalar; it has no components to it. In addition to giving us the area of a parallelogram, determinants also allow us to find solutions to linear equations. This property is known as Cramer's Law:

$$\begin{aligned} ax+by+cz&=d \\ ex+fy+gz&=h \\ ix+jy+kz&=l \end{aligned} \implies x = \frac{\begin{vmatrix} d & b & c \\ h & f & g \\ l & j & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & l & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}}; \quad z = \frac{\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}}$$

The solution for x is the determinant of the matrix with its x variables replaced by the solutions of each equality divided by the determinant of the coefficient matrix. It is important to note that in order for Cramer's rule to work, the determinant of the coefficient matrix cannot equal zero, because the denominator would be zero. The proof for this is rigorous, so it won't be included in this text.

Finding the determinant of a 3×3 matrix can be very arduous. Although we can find the determinant by using the previous method, there is another method: expansion by minors. If we wish to find the determinant of the 3×3 matrix A, then we could use the diagonal method (drawing diagonal lines through the constants in the matrix should make elucidate why it is called the diagonal method).

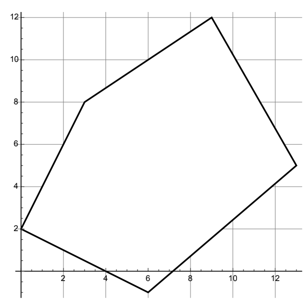
$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

However, this can become quite taxing when dealing with larger numbers. When dealing with large numbers or large matrices (generally, anything larger than a 3×3 matrix), expansion by minors is the way to go. The rule states:

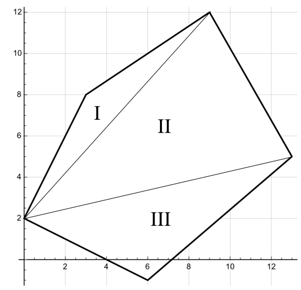
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Another way to think about this is that for each constant in the top row of the matrix, it is multiplied by the determinant of the constants that are not in the same row or column as that particular constant. A good way to go about this is to cross out the row and column that the variable is in, and then write the new determinant of the non-crossed variables.

One useful application of determinants is that they allow us to find the area of a polygon. Consider a pentagon with vertices $(0,2), (3,8), (9,12), (13,5)$, and $(6,-1)$:



Although there is no easy formula to find the area of an n-sided polygon with matrices, there is a clever thing we can do to find the area of a polygon like this; we can split it up into different triangles and then take the area of each triangle. One way the pentagon can be split up is like this:



Starting off with the area of triangle I, we have the vertices $(0,2),(3,8)$, and $(9,12)$. Since the determinant gives us the parallelogram formed by two sides, we need half of the determinant to give us the area of the triangle (because half of a parallelogram is a triangle). Therefore, we can set up the equation:

$$A_{\text{I}} = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 8 & 1 \\ 9 & 12 & 1 \end{vmatrix}$$

The third column is all ones for two main reasons - we want to keep the triangle in one dimension, and a determinant cannot be taken from a non-square matrix. Although it can be proven, it is an extremely long proof, so it will not be included in this. After evaluating the determinant to find A_{I} (hopefully by expansion by minors), we find that A_{I} is $-12u^2$, where u represents any arbitrary unit (this could be kilograms, meters, seconds, etc). However, since we are working with units of area, then it obviously could only be meters, feet, or any other unit of length). Although A_{I} gives a negative value, we can negate the minus sign since we are working with area, which is always positive. After finding A_{I}, we can go on to find A_{II} and A_{III}:

$$A_{\text{II}} = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 9 & 12 & 1 \\ 13 & 5 & 1 \end{vmatrix}; \quad A_{\text{III}} = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 13 & 5 & 1 \\ 6 & -1 & 1 \end{vmatrix}$$

$$A_{\text{II}} = 51.5u^2; \; A_{\text{III}} = 28.5u^2$$

Therefore, the total area is $A_{\text{I}} + A_{\text{II}} + A_{\text{III}} = 92u^2$

Determinants also have numerical properties. For example, if we are evaluating the determinant of matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ it is not the same as the determinant for matrix $B = \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{bmatrix}$. Instead, the determinant for B is k times that of A. Our first determinant property is:

$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

This is pretty easy to prove as well; if we take the determinant of matrix B, by expansion of minors, we get

$$ka(ei-hf) - kb(di-gf) + kc(dh-eg) = k[a(ei-hf) - b(di-gf) + c(dh-eg)]$$

Then, if we take the determinant of matrix A, we get

$$a(ei-hf)-b(di-gf)+c(dh-eg)$$

This shows that the determinant of matrix B is k times that of matrix A.
In addition, if we were to interchange the rows of a matrix, the determinant would be multiplied by a factor of -1. The proof for this is pretty straightforward as well:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf)-b(di-gf)+c(dh-eg)$$
$$= aei-ahf-bdi+bgf+cdh-ceg = \alpha$$

The purpose of α, or alpha, is simply to consolidate the left side of the equation into one quantity. Switching the first and second rows,

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = d(bi-hc)-e(ai-gc)+f(ah-gb)$$
$$= dbi-dhc-eai+egc+fah-fgb$$
$$= -(aei-ahf-bdi+bgf+cdh-ceg) = -\alpha$$

Logically, if we were to switch one row with another, and then another with another, the determinant would remain $(-1*-1=1)$.
Finally, if we add one row to another row, it has no effect on the determinant. Namely,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{vmatrix}$$, where k is a constant.

The proof:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei-ahf-bdi+bgf+cdh-ceg = \alpha$$

$$\begin{vmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{vmatrix} = (a+kd)(ei-hf)-(b+ke)(di-gf)+(c+kf)(dh-eg)$$
$$= aei-ahf+kdei-kdhf-bdi+bgf-kedi+kegf+cdh-ceg+kfdh-kfeg$$

Note how all the terms with four variables cancel out.

$$= aei-ahf-bdi+bgf+cdh-ceg = \alpha$$

.
To summarize,

- Multiplying a row by constant k multiplies the determinant by k
- Switching two rows multiplies the determinant by -1
- Adding one row to another does not change the determinant

Example 3.2

If the area of the triangle formed by the points $(0,0)$, $(4,2)$, and $(1,5)$ is $9u^2$, what is the area of the triangle formed by the points $(2,1)$, $(22,3)$, and $(7,6)$?

The purpose of this problem is to utilize the properties of determinants, rather than plot the triangle and compute its area. What we are given is the area of the triangle formed by three points. Putting this in determinant form,

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 1 & 5 & 1 \end{vmatrix} = 9$$

Now, if we look closely at these points, we can notice that each point is related to the original point: $(x,y) \to (5x+2, y+1)$. If we rewrite our original coordinates as (x_1,x_1), (x_2,y_2), and (x_3,y_3),

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 9$$

We want to find

$$\frac{1}{2}\begin{vmatrix} 5x_1+2 & y_1+1 & 1 \\ 5x_2+2 & y_2+1 & 1 \\ 5x_3+1 & y_3+1 & 1 \end{vmatrix}$$

As it turns out, because of the symmetry of the determinant, multiplying a column by a constant k is also multiplies the determinant by k. Additionally, adding a constant to a column has no effect on the determinant. Because of this,

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} 5x_1+2 & y_1+1 & 1 \\ 5x_2+2 & y_2+1 & 1 \\ 5x_3+2 & y_3+1 & 1 \end{vmatrix} &= 5 * \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ &= 5 * 9 \\ &= \boxed{45} \end{aligned}$$

3.7 Inverse Matrices

An inverse matrix is a matrix that, when multiplied by its counterpart, produces the identity matrix. An identity matrix has ones in its diagonal and zeroes everywhere else. For example, a 3×3 identity matrix would look like:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the inverse of matrix A is A^{-1}, then $A \times A^{-1} = I$, where I is the identity matrix. The inverse of any square matrix can be found by using the formula:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) \tag{27}$$

The "adj" means the adjugate of a matrix, which is quite similar to the expansion of minors method:

$$\text{If } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then } \operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Note the sign pattern that the adjugate matrix has: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ Similarly, if the matrix were 2×2, the adjugate would have the signs $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$.

It is important to note that if the determinant of a matrix is zero, than there is no inverse matrix. If the determinant is zero, than each row in the matrix is linearly dependant of the others; namely, at least one row is a scalar multiple of another row in the matrix. Thinking about this in a coordinate system, the area of our parallelogram would be zero, because the two points are linear with the origin. In addition, considering the equation shown above, $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$, A^{-1} cannot exist if the determinant is zero because the denominator would be zero.

Example 3.3

Find the inverse matrix of $A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

We are going to use the formula $A^{-1} = \frac{1}{\det A}\,\text{adj}(A)$ to find A^{-1}. We can start by finding $|A|$:

$$\begin{aligned}\begin{vmatrix} 3 & -4 \\ 2 & 2 \end{vmatrix} &= 3(2) - 2(-4) \\ &= 14\end{aligned}$$

Now, we can find adj(A):

$$\text{adj}(A) = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

Thus,

$$\begin{aligned} A^{-1} &= \frac{1}{\det A}\,\text{adj}(A) \\ &= \frac{1}{14}\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1/7 & 2/7 \\ -1/7 & 3/14 \end{bmatrix}\end{aligned}$$

In addition to the adjugate method of finding the inverse of a matrix, there is a method similar to an augmented matrix. If we want to find the inverse of the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, we can make an augmented matrix and set it equal to its corresponding identity matrix:

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

We then get the left side in reduced row echelon form via row operations, and the right side of the matrix should be the identity matrix.

Example 3.4

Find the inverse of $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$.

We will solve this by using the method described above (this will require a lot of row operations):

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \Rightarrow R_1 = -r_1; R_2 = -r_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow R_2 = -r_2 - r_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow R_3 = -r_3 - 6r_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 4 & -3 & 6 & 0 & 1 \end{array}\right]$$

$$\Rightarrow R_3 = -r_3 - 4r_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right] \Rightarrow R_2 = r_2 + r_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right]$$

$$\Rightarrow R_1 = r_1 + r_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right] \Rightarrow \text{The inverse matrix is } \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

This method may seem confusing at first, but with practice it becomes much easier. One of the biggest problems with this method is that few people understand why it actually works. The main thing to realize in this process is that the product of a matrix and its inverse is always an identity matrix. In addition, the product of any matrix and its identity matrix is the same matrix. Given a square matrix A, we set it equal to the identity matrix:

$$\left[A \mid I\right]$$

This is just a simpler version of an augmented matrix.
Then, we get the left side to be an identity matrix, which is also $A^{-1}A$. This is equivalent to multiplying both sides by A^{-1}:

$$\left[A^{-1}A \mid IA^{-1}\right]$$

Simplifying, we get:

$$\left[I \mid A^{-1}\right]$$

This takes advantage of the fact that $A^{-1}A = I$ and $A^{-1}I = A^{-1}$.
As you've probably realized by now, there is no easy way to find the inverse of a 3×3 matrix. Unless you have a calculator, either one of these two methods will require a lot of time and work. The best way to reduce the time spent on one of these is to practice a lot and get comfortable with both methods.

Another application of inverse matrices is that they allow us to find solutions to systems of equations. For example, if we have three linear equations with three variables, we can put them into a coefficient matrix and solve for each variable. Consider the three equations

$$\begin{aligned} 3x + 4y - 7z &= 2 \\ 2x - y + 3z &= 11 \\ -5x + 3y - z &= 6 \end{aligned}$$

We can then set up an equation to solve for *x, y* and *z*:

$$\begin{bmatrix} 3 & 4 & -7 \\ 2 & -1 & 3 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 6 \end{bmatrix}$$

When we multiply the two matrices on the left side, we end up with our original system of equations; try this for yourself if this step is not clear.
Using a calculator, we can easily find the inverse matrix:

$$\begin{bmatrix} .096 & .205 & -.060 \\ .157 & .458 & .277 \\ -.012 & .349 & .133 \end{bmatrix}$$

We can multiply both sides by the inverse matrix to get

$$\begin{bmatrix} .096 & .205 & -.060 \\ .157 & .458 & .277 \\ -.012 & .349 & .133 \end{bmatrix} \begin{bmatrix} 3 & 4 & -7 \\ 2 & -1 & 3 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .096 & .205 & -.060 \\ .157 & .458 & .277 \\ -.012 & .349 & .133 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ 6 \end{bmatrix}$$

Since the left side is a matrix multiplied by its inverse, the product is the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .096 & .205 & -.060 \\ .157 & .458 & .277 \\ -.012 & .349 & .133 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .096 & .205 & -.060 \\ .157 & .458 & .277 \\ -.012 & .349 & .133 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ 6 \end{bmatrix}$$

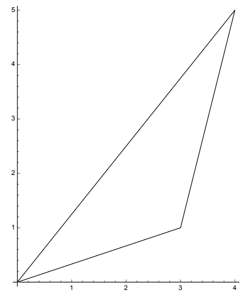
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2.084 \\ 7.012 \\ 4.614 \end{bmatrix}$$

Therefore, $x = 2.084$, $y = 7.012$, and $z = 4.614$, or, to be more exact, $\frac{173}{83}$, $\frac{582}{83}$, and $\frac{383}{83}$.

3.8 Transformation Matrices

One of the many applications of matrices is that they can be used to represent the transformation of a set of points. There are four main transformations we will focus on: translations, dilations, reflections, and rotations.

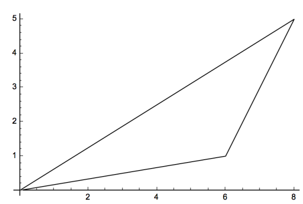
Consider a triangle represented by the points $(0,0),(3,1)$, and $(4,5)$. The points can be represented by the matrix $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}$. It will become more clear later why we arrange ordered pairs vertically when dealing with transformations. When graphed, the shape would appear as:



Now, assume we want to horizontally dilate, or stretch, the triangle by a factor of 2. To be more specific, we want to multiply the x-coordinate of every point in the triangle by 2, while keeping the y-coordinates the same. Surprisingly enough, a matrix can be multiplied to our original matrix to create a new set of points. We call this new, multiplied matrix a transformation matrix. In this case, our transformation matrix,T, is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Thinking about why this is our matrix, it makes sense. Multiplying T to A will produce a 2×3 matrix, and each x coordinate will be multiplied by 2, while the y coordinate will be multiplied 1 and remain the same.

$$TA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2(0)+0(0) & 2(3)+0(1) & 2(4)+0(5) \\ 0(0)+1(0) & 0(3)+1(1) & 0(4)+1(5) \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 5 \end{bmatrix}$$

It is important to note that the transformation matrix comes before the point matrix. After applying the transformation, we have a new triangle:

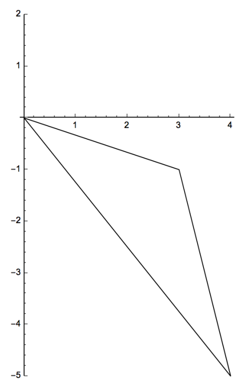


More generally, if we want to dilate any shape by a factor of k, the transformation matrix is $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ if we are dilating the x-coordinates. If we are dilating the y-coordinates, the transformation matrix is $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

In addition to transformation matrices that dilate an image, we can also reflect an image using a matrix. This is done simply by multiplying the x or y coordinates of a set of points by -1. If we wanted to reflect the triangle over the x-axis, we would multiply all the y-coordinates of the image by -1. Thus, a transformation matrix can be set up with relative ease: $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ We can then multiply T to A to get a new set of points, or A'.

$$TA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1(0)+0(0) & 1(3)+0(1) & 1(4)+0(5) \\ 0(0)+(-1)(0) & 0(3)+(-1)(1) & 0(4)+(-1)(5) \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

The resulting image, when graphed:



If we wanted to reflect the image over the y-axis, we would have to multiply all the x-coordinates by a factor of -1. The transformation matrix would be $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, assume we wanted to rotate the image. As you may vaguely remember, if we want to rotate an image 90° clockwise, we switch the x and y-coordinates and multiply the y by a factor of -1. The resulting matrix would be $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Notice how the x and y have switched and how the y, or the bottom row of the matrix, has been multiplied by -1. By the same idea, if we wanted to rotate an image 90° counterclockwise, our transformation matrix would be $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; we are switching the x and y and multiplying the x by -1. This is analogous to the transformation $(x,y) \to (-y,x)$.

What if we want to rotate an image 30° counterclockwise? As you may have noticed, the solution to this is not as intuitive as the previous one. Rather, our transformation matrix is

$$T_{\text{counter}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{28}$$

As it turns out, this is the transformation matrix for any counterclockwise rotation.

For any clockwise rotation, $T_{\text{clockwise}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. This makes sense, because a clockwise rotation of angle θ is the same as a counterclockwise rotation of $360° - \theta$, which just $-\theta$. Because of this, if we were to represent the clockwise rotation of angle θ in a counterclockwise transformation matrix, it would be $T = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$. Using trigonometric identities, we get our clockwise transformation matrix:

$$T_{\text{clockwise}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{29}$$

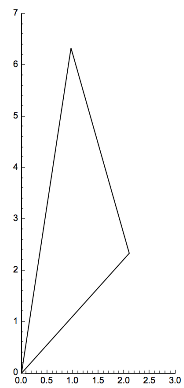
Therefore, the transformation matrix for a 30° counterclockwise rotation would be

$$T = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Multiplying T to A, we get

$$TA = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 3\sqrt{3}/2 - 1/2 & 4\sqrt{3}/2 - 5/2 \\ 0 & 3/2 + \sqrt{3}/2 & 2 + 5\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 0 & 2.098 & 0.946 \\ 0 & 2.336 & 6.330 \end{bmatrix}$$

Our new image, graphed:



What if we wanted to translate the triangle 3 units up and 1 unit to the right? We could just add a translation matrix to the point matrix, but there is a more useful and efficient way of doing this. If we want to translate an image vertically a units and horizontally b units, then our transformation matrix is

$$T = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

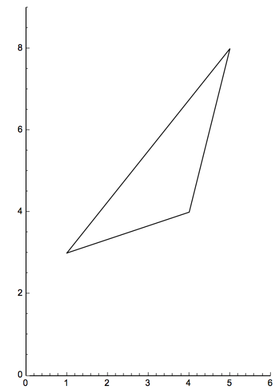
Although this may seem confusing, it actually works out algebraically; given a set of points (x_1, y_1), (x_2, y_2), and (x_3, y_3), we can put the coordinates into the matrix $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}$. The bottom row of ones simply makes the matrix multiplication possible.

$$TA = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 + a & x_2 + a & x_3 + a \\ y_1 + b & y_2 + b & y_3 + b \\ 1 & 1 & 1 \end{bmatrix}$$

So, if we wanted to translate the triangle 3 units up and 1 unit to the right, our transformation matrix would be $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1 & 3+1 & 4+1 \\ 0+3 & 1+3 & 5+3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 4 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

.



When multiple transformations are performed on a set of points, we can multiply transformation matrices together to get one composite transformation matrix. It is important to note that the order in which we multiply matrices together does matter (the associative property does not apply to matrices), so the transformation that is being applied first should be the one closest to the point matrix. For example, if we are performing transformations *A, B, C,* and *D* (in that specific order) to a matrix of points *P,* then the correct order of matrix multiplication would be *DCBAP*.

3.9 Problems

1. Solve the following systems of equations using Gauss-Jordan elimination:

   (a) $$\begin{aligned} 3x+y+2z&=5 \\ 5x+10y-5z&=4 \\ -x+y-z&=2 \end{aligned}$$

   (b) $$\begin{aligned} x-2y-6z&=12 \\ 2x+4y+12z&=-17 \\ x-4y-12z&=22 \end{aligned}$$

   (c) $$\begin{aligned} 2x+11y+5z&=9 \\ 2x+8y-4z&=0 \\ 4x+18y+3z&=11 \end{aligned}$$

   (d) $$\begin{aligned} {}^3/x+{}^4/y&=-1 \\ {}^1/x+{}^2/y&=5 \end{aligned}$$

2. Given the following three matrices, state whether the following matrix products can be found. If possible, find the product.

$$A=\begin{bmatrix} 3 & -2 & -4 & 2 \\ -1 & 7 & 6 & 2 \\ 4 & 1 & -3 & 5 \end{bmatrix} \quad B=\begin{bmatrix} 5 & 6 & 4 \\ -2 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad C=\begin{bmatrix} -3 & 6 & -1 & 4 \\ 5 & 1 & 5 & 3 \end{bmatrix}$$

$$D=\begin{bmatrix} -4 & 5 \\ 3 & 0 \\ -3 & 3 \end{bmatrix} \quad E=\begin{bmatrix} -2 & 6 & -1 \\ -7 & -7 & 3 \\ 4 & -7 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

   i. ED
   ii. DC
   iii. CE
   iv. AE
   v. AC

3. Find the area of a hexagon with vertices $(-2,0)$, $(-1,3)$, $(3,4)$, $(5,3)$, $(4,0)$, and $(2,-1)$.

4. Find the equation of a polynomial that passes through the points $(-2,43)$, $(-1,-2)$, $(0,-1)$, $(1,2)$, and $(2,-23)$. Assume the least degree possible.

5. Given lines $y=2x+\frac{1}{2}$ and $y=-\frac{1}{2}x+5$, find the area enclosed by the lines, the x-axis, and the y-axis.

6. Find the inverses of the following matrices

(a) $\begin{bmatrix} 4 & 5 & 5 \\ -2 & -3 & -4 \\ 1 & 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} -4 & -1 & 0 \\ 2 & 2 & -5 \\ -3 & -5 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 4 & 3 \\ -4 & 4 & -4 \end{bmatrix}$

7. Let $P = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ be a matrix of points. List the matrices to perform each of the transformations and find the matrix for the points of the new image.
   (a) Reflect over the y-axis
   (b) Dilate by a factor of 2 in both directions
   (c) Translate 2 units to the left and 3 units down.

8. Use matrices to find the area in the first enclosed by the lines $y = \frac{1}{4}x + 4$ and $y = 2x - 3$.

9. Khaleesi is experimenting with different foods for her dragons. The nutritional information per ounce of each ingredient is given in the first matrix. The percentage of each ingredient in each mix are given in the second matrix, and the calories per macronutrient (fat, protein, carbohydrate) are given in the third, C.

	Sheep	Cow	Bear	Goat
Fat	6g	0.2g	1g	0.5g
Protein	3g	0.8g	3g	2g
Carbs	2g	22g	23g	23g

Calories per Gram

Fat	9
Protein	4
Carbs	4

	Mix A	Mix B
Peanuts	30%	40%
Raisins	20%	10%
Pretzels	25%	20%
Chex	25%	30%

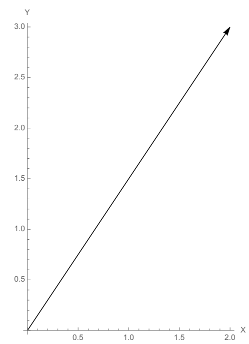
Set up and compute a system of matrices that will determine the number of calories per gram for each dragon meal.

4 Vectors

This chapter will lay the basic foundation for classical physics. A vector is defined as a quantity that has both magnitude and direction. So, what does this mean? In two dimensions, a vector has two components - a horizontal and vertical component. It's tail starts at its initial point, and has a head which points to the terminal point. Vectors can be extremely useful for representing the position of a particle, as well as its velocity, acceleration, and many other quantities.

4.1 Component Form

Vectors can be represented in many different says. We will start off with component form. This simply lists each component of the vector, enclosed by carrots. The general form for a vector, let's use $\vec{a}$, in its component form, is $\vec{a} =< x,y,z >$. For example, consider the vector $\vec{a} =< 2,3 >$. It has two components, an x and a y. The x component is 2, and the y component is 3.



The magnitude of the vector is its length. Namely, it is the square root of the sum of the squares of the components. If we are showing the magnitude of a vector, say $\vec{a}$, then we can write it as either A, $||\vec{a}||$, or $|\vec{a}|$.

$$\text{If } \vec{a} =< x,y,z >, \text{ then } |\vec{a}| = \sqrt{x^2+y^2+z^2}$$

Example 3.1

If a 2-dimensional vector has a magnitude of 3 and a horizontal component of 2, find its vertical component.

We know that the horizontal component, or x component of the vector is 2. We also know that the magnitude is 3. Using the equation above, $3 = \sqrt{2^2+y^2}$. With algebra, we find that the vertical, or y component, is $\sqrt{5}$.

4.2 Linear-Combination form

Okay, so pretty simple, right? Now, let's talk about another form of vector notation, called Linear-Combination form. First, we have to understand what a **unit vector** is. A unit vector is any vector with a magnitude of 1. As you will later learn, unit vectors, are, ironically, unitless. All they give you is the direction of a vector which points in the same direction, but has a magnitude that is not 1. In addition, when writing unit vectors, we put a "hat" above them to signify that they are unit vectors. With that being said, three new unit vectors will be introduced: $\hat{i}$, $\hat{j}$,and $\hat{k}$. Each of the unit vectors points in a different direction. $\hat{i}$ points in the +x-direction, $\hat{j}$ points in the +y-direction, and $\hat{k}$ points in the +z-direction. We call these three the **principal unit vectors**.

Example 3.2:

Find the component forms of each of each of the principal unit vectors

Each of the vectors is a unit vector, meaning it has a magnitude of 1. Since each of the principal points in a unique coordinate direction, $\hat{i} =< 1,0,0 >$, $\hat{j} =< 0,1,0 >$, and $\hat{k} =< 0,0,1 >$.

Linear-Combination form consists of different scalars multiplied by their corresponding principal unit vector. For example, if the component form of a vector is $< 3,2,4 >$, then its Linear-Combination form is $3\hat{i}+2\hat{j}+4\hat{k}$.

4.3 Magnitude-Direction form

Although there are many other ways to represent vectors, we will only focus on one more form: **Magnitude-Direction form**. The name says it all. Enclosed by brackets, the vector is represented by its magnitude, followed by its direction. To be specific about direction, this is the counter-clockwise angle the vector makes with the +x axis. For example, the Magnitude-Direction form of $< 1,1 >$ is $[\sqrt{2},45°]$. Magnitude-Direction form is especially useful when trying to figure out the forces acting on an object and where it will go, but it is hard to compare it to other vectors without putting it into its components. Converting a vector in Magnitude-Direction form into its component form is intuitive:

$$\vec{a} = [|\vec{a}|,\theta] =< |\vec{a}|\cos\theta, |\vec{a}|\sin\theta >= |\vec{a}|\cos\theta\hat{i} + |\vec{a}|\sin\theta\hat{j} \tag{31}$$

$$a_x = |\vec{a}|\cos\theta \qquad a_y = |\vec{a}|\sin\theta \tag{32}$$

Where a_x and a_y are the x and y components, respectively of vector $\vec{a}$.
In addition, if we want to find the magnitude of a vector given its components, we can use the distance formula: given a vector $\vec{a}$ with its tail at the origin and head at point (x,y) the magnitude is

$$|\vec{a}| = \sqrt{x^2+y^2} \tag{33}$$

More generally, a vector with its tail at the point (x_0,y_0) and head at point (x_1,y_1) has a magnitude

$$|\vec{a}| = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} = \sqrt{(\Delta x)^2+(\Delta y)^2} \tag{34}$$

4.4 Properties of Vectors

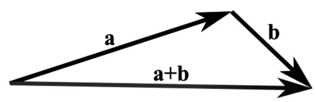
Vectors have certain properties which become necessary when doing any computation with them. First, in order to add vectors, the individual components must be added together. Namely, given vectors $\vec{a} = a_x\hat{i} + a_y\hat{j}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j}$,

$$\vec{a} + \vec{b} = (a_x\hat{i} + a_y\hat{j}) + (b_x\hat{i} + b_y\hat{j}) = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} \tag{35}$$

Similarly, with subtraction:

$$\vec{a} - \vec{b} = (a_x\hat{i} + a_y\hat{j}) - (b_x\hat{i} + b_y\hat{j}) = (a_x - b_x)\hat{i} + (a_y - b_y)\hat{j} \tag{36}$$

Vectors can also be added together graphically using the head-to-tail method. This method involves graphing the first vector and then the second, but with its tail on the head of the first vector:



In addition, vectors can be multiplied by a scalar. If $\vec{a} = a_x\hat{i} + a_y\hat{j}$, we can multiply it by a scalar β to get

$$\beta\vec{a} = \beta a_x\hat{i} + \beta a_y\hat{j} \tag{37}$$

This also multiplies the magnitude of $\vec{a}$ by β as well:

$$|\beta\vec{a}| = \sqrt{(\beta a_x)^2 + (\beta a_y)^2} = \sqrt{\beta^2(a_x^2 + a_y^2)} = \beta\sqrt{a_x^2 + a_y^2} = \beta|\vec{a}|$$

Any vector can also be converted into a unit vector:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \tag{38}$$

This makes sense. If we divide every component of $\vec{a}$ by its magnitude, then the resulting magnitude will be 1, making a unit vector in the direction of $\vec{a}$:

$$\frac{\vec{a}}{|\vec{a}|} = \frac{a_x}{|\vec{a}|}\hat{i} + \frac{a_y}{|\vec{a}|}\hat{j}$$

Which we can rewrite and simplify as

$$\sqrt{\left(\frac{a_x}{|\vec{a}|}\right)^2 + \left(\frac{a_y}{|\vec{a}|}\right)^2} = \sqrt{\left(\frac{1}{|\vec{a}|}\right)^2(a_x^2 + a_y^2)} = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

4.5 Dot Product

The dot product of two vectors quantifies how much one vector goes in in the direction of the other. In addition, it gives us a a sense of the angle between the two vectors; this will become more clear later. Contrary to most of the other quantities in this chapter, the dot product is a scalar, meaning it has no dimensions. We denote the dot product operator with a simple •. Given vectors $\vec{a} = a_x\hat{i} + a_y\hat{j}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j}$,

$$\vec{a} \bullet \vec{b} = a_x b_x + a_y b_y \tag{39}$$

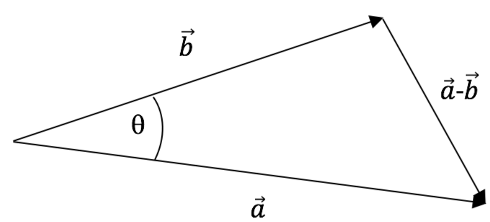
There are several properties of the dot product as well, all of which can be proven algebraically by separating the vectors into component form:

$$\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$$

$$\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$$

$$\vec{a} \bullet \vec{a} = |\vec{a}|^2$$

There is also an identity of the dot product that gives us the exact angle between two vectors. Consider the figure below.



Recall the Law of Cosines (page 11)

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

In addition, because of dot product properties,

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \bullet (\vec{a} - \vec{b}) \\ &= \vec{a} \bullet \vec{a} - \vec{a} \bullet \vec{b} - \vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{b} \\ &= |\vec{a}|^2 - 2(\vec{a} \bullet \vec{b}) + |\vec{b}|^2 \end{aligned}$$

Because $|\vec{a}|^2 - 2(\vec{a} \bullet \vec{b}) + |\vec{b}|^2$ and $|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$ are both equal to $|\vec{a} - \vec{b}|^2$,

$$|\vec{a}|^2 - 2(\vec{a} \bullet \vec{b}) + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

Simplifying,

$$\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\bullet\vec{b}}{|\vec{a}||\vec{b}|}\right) \tag{40}$$

This is arguably the most important dot product identity; it allows us to compute the angle between two vectors simply off of their magnitudes and their dot product. In addition, it gives us an important property of the dot product: if the dot product is zero, then the vectors are orthogonal, or perpendicular, to each other. This makes sense; if $\vec{a}\bullet\vec{b} = 0$, than $\theta = \cos^{-1}\left(\frac{0}{|\vec{a}||\vec{b}|}\right) = \cos^{-1} 0 = \frac{\pi}{2}$

Example 3.3

Find the angle between vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$ and $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

We will use $\theta = \cos^{-1}\left(\frac{\vec{a}\bullet\vec{b}}{|\vec{a}||\vec{b}|}\right)$ to solve this problem. Starting off, we can find $\vec{a}\bullet\vec{b}$:

$$\begin{aligned}\vec{a}\bullet\vec{b} &= (3*2) + (-2*4) + (5*-2) \\ &= -12\end{aligned}$$

Now, finding $|\vec{a}|$ and $|\vec{b}|$,

$$\begin{aligned}|\vec{a}| &= \sqrt{(3)^2 + (-2)^2 + (5)^2} \\ &= \sqrt{38}\end{aligned}$$

$$\begin{aligned}|\vec{b}| &= \sqrt{(2)^2 + (4)^2 + (-2)^2} \\ &= \sqrt{24}\end{aligned}$$

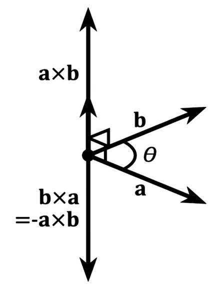
Now, we can use the formula:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{a}\bullet\vec{b}}{|\vec{a}||\vec{b}|}\right) \\ &= \cos^{-1}\left(\frac{-12}{\sqrt{38*24}}\right) \\ &= \boxed{113^\circ}\end{aligned}$$

4.6 Cross Product

The cross product of two vectors gives a vector perpendicular to the two with a magnitude of the parallelogram formed by the vectors (or the determinant of the two vectors). In addition, the cross product has numerous applications in physics, specifically with rotational dynamics; the vector formed by the cross product is the axis of rotation. The plane formed by the two vectors is also known as the plane of rotation.

We denote the cross product operator as a $\times$. For example, the cross product of vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$. Calculating the cross product involves setting up a matrix and using a method similar to expansion by minors:



If $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ and $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\hat{\imath} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\hat{k}$$

Another way to think about the cross product is as one big determinant:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

It is important to note that $\vec{a}\times\vec{b}$ is not the same as $\vec{b}\times\vec{a}$. Rather, they are opposite vectors with the same magnitude:

$$\vec{a}\times\vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\hat{\imath} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\hat{k} = (a_yb_z - b_ya_z)\hat{\imath} - (a_xb_z - b_za_z)\hat{\jmath} + (a_xb_y - b_xa_y)\hat{k}$$

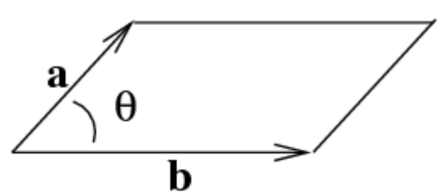
$$\vec{b}\times\vec{a} = \begin{vmatrix} b_y & b_z \\ a_y & a_z \end{vmatrix}\hat{\imath} - \begin{vmatrix} b_x & b_z \\ a_x & a_z \end{vmatrix}\hat{\jmath} + \begin{vmatrix} b_x & b_y \\ a_x & a_y \end{vmatrix}\hat{k} = (b_ya_z - a_yb_z)\hat{\imath} - (b_xa_z - a_zb_z)\hat{\jmath} + (b_xa_y - a_xb_y)\hat{k} = -\vec{a}\times\vec{b}$$

Thus,

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \tag{41}$$

Similar to the dot product, the cross product also describes a feature of the two vectors if it is zero: the vectors are parallel. This can be explained a few ways. First, since the cross product is essentially a determinant, and because parallel vectors are different by only a scalar, two rows of the determinant will not be linearly independent. Therefore, the determinant, or cross product, will be zero. Another way to think about this is the area of the parallelogram formed by the two vectors. Remember how the magnitude of the cross product vector is equal to the area of the parallelogram formed by the two vectors. Since the area of a parallelogram formed by two parallel vectors is zero, the magnitude of the cross product is zero.

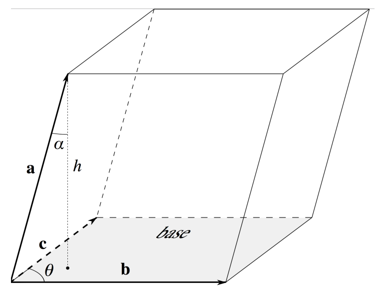
There is another property we can get out of the cross product. Think about a parallelogram formed by two vectors:



We know the area of the parallelogram is (base)(height), or $|\vec{b}||\vec{a}|\sin\theta$. The height is the y-component of $\vec{a}$, or $|\vec{a}|\sin\theta$. If this isn't obvious to you, prove it with a right triangle! In addition, we know that the area of the parallelogram is the magnitude of the cross product vector, or $|\vec{a}\times\vec{b}|$. Therefore, we have the equality:

$$|\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}| \tag{42}$$

This equality leads us to our next application of cross products; they play a role in finding the volume of a parallelepiped. Basically, a parallelepiped is just a parallelogram extruded to the third dimension.



The volume of a parallelepiped is given by the equation (A_{base})(height). Translating this into vector language,

$$\begin{aligned} V &= (A_{\text{base}}) \\ &= |\vec{b}\times\vec{c}||\vec{a}|\cos\alpha \\ &= |\vec{b}||\vec{c}|\sin\theta|\vec{a}|\cos\alpha \end{aligned}$$

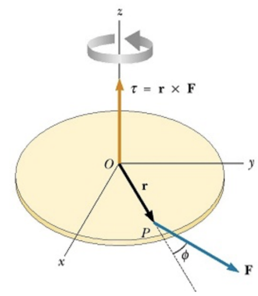
Now, recall from the dot product that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors (see figure). The equation $V = |\vec{b}\times\vec{c}||\vec{a}|\cos\alpha$ is the same thing in disguise. As it turns out, the angle between the cross product vector $\vec{b}\times\vec{c}$ and $\vec{a}$ is actually α. This works out beautifully geometrically; $\vec{b}\times\vec{c}$ is perpendicular to the plane formed by vectors $\vec{b}$ and $\vec{c}$. Because of alternate interior angles, the angle between $\vec{b}\times\vec{c}$ and $\vec{a}$ is actually the same as the angle between $\vec{a}$ and $\vec{h}$; α. We get our final formula for the volume of a parallelepiped:

$$V = \vec{a}\cdot(\vec{b}\times\vec{c}) \tag{43}$$

This is equivalent to $|\vec{b}\times\vec{c}||\vec{a}|\cos\alpha$.
Finally, the last cross product application we will cover is torque. Torque is defined as the rate of change of angular momentum of an object, or a measure of the rotational force on an object. Although you don't need to have a full understanding of torque until you take a physics course, the calculation of it is very doable with vector concepts. Torque, τ, is mathematically defined as

$$\vec{\tau} = \vec{r}\times\vec{F} \tag{44}$$

Where $\vec{r}$ is the distance from the axis of rotation to the object and $\vec{F}$ is the force applied to the object. One of the most non-intuitive parts of torque is that it is a vector **perpendicular** to the plane of rotation. This is because the direction of the torque gives us a sense of how the object is rotating (clockwise or counterclockwise). The direction of the rotation (and thus, torque) can be calculated using the the right hand rule. Given vectors $\vec{a}$ and $\vec{b}$, the direction of the cross product $\vec{a} \times \vec{b}$ is found by pointing your right hand in the direction of $\vec{a}$ and curling it in the direction of vector $\vec{b}$. Then, stick your thumb out. The direction that your thumb is pointing is the direction of the torque vector.

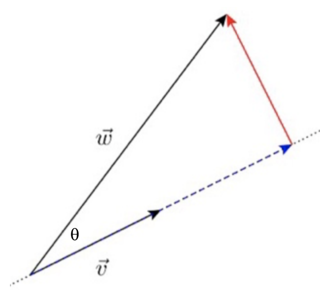


4.7 Projections

Projections, in short, tell us how much of a vector goes in the same direction of another vector. For example, if we project the vector $\vec{a} = 3\hat{i} + 2\hat{j}$ onto the x-axis, we will get a magnitude of 3. This is because $\vec{a}$ goes 3 units in the same direction as the x axis. Similarly, given magnitude direction form of a vector $\vec{a}$, we can project it onto the x and y-axis using trigonometry:

$$\begin{aligned}[|\vec{a}|, \theta] &= (|\vec{a}|\cos\theta)\hat{i} + (|\vec{a}|\sin\theta)\hat{j} \\ &= \text{proj}_{x-\text{axis}}\vec{a} + \text{proj}_{y-\text{axis}}\vec{a}\end{aligned}$$

But what if we want to project a vector onto something other than the simple x and y-axes? This is where it gets complicated. Say we want to find the projection of an arbitrary vector $\vec{w}$ onto vector $\vec{v}$:



Right off the bat, a few things become obvious: the length of $\vec{w}$ in the same direction as $\vec{v}$ is $|\vec{w}|\cos\theta$. In addition, we know that $\cos\theta = \frac{\vec{v}\bullet\vec{w}}{|\vec{v}||\vec{w}|}$. Then, we substitute in for $\cos\theta$ to get the length $\vec{w}$ in the direction of $\vec{v}$:

$$|\vec{w}|\frac{\vec{v}\bullet\vec{w}}{|\vec{v}||\vec{w}|} = \frac{\vec{v}\bullet\vec{w}}{|\vec{v}|}$$

This is the *magnitude* of the vector in the same direction as $\vec{v}$. However, if we want to get it to be an actual vector, all we have to do is multiply the scalar by $\hat{v}$; recall that all a unit vector gives is the direction. We then get $\text{proj}_{\vec{v}}\vec{w}$, the projection of $\vec{w}$ onto $\vec{v}$:

$$\begin{aligned}\text{proj}_{\vec{v}}\vec{w} &= \frac{\vec{v}\bullet\vec{w}}{|\vec{v}|}\hat{v}\\ &= \frac{\vec{v}\bullet\vec{w}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|}\\ &= \frac{\vec{v}\bullet\vec{w}}{|\vec{v}|^2}\vec{v}\end{aligned}$$

Our projection of a vector onto the x-axis is just a simplified version of this formula: given vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$, it is logical that the projection of $\vec{a}$ onto the x-axis is $a_x\hat{\imath}$. When we use the formula (we will use as the x-axis vector, or what $\vec{a}$ is being projected onto) for the projection of $\vec{a}$ onto $\hat{\imath}$, we get:

$$\begin{aligned}\text{proj}_{\hat{\imath}}\vec{a} &= \frac{\vec{a}\bullet\hat{\imath}}{|\hat{\imath}|^2}\hat{\imath}\\ &= [(a_x\hat{\imath} + a_y\hat{\jmath})\bullet\hat{\imath}]\hat{\imath}\\ &= a_x\hat{\imath}\end{aligned}$$

This is because $\hat{\imath}\bullet\hat{\jmath} = 0$ (because they are perpendicular) and because $|\hat{\imath}| = 1$, so $|\hat{\imath}|^2 = 1$

4.8 3-Dimensional Equations

It is somewhat intuitive to think that given three non-linear points in space, one unique plane goes through all of those points. This assumption is true, but how do we find the equation of the plane? As it turns out, we can use vectors to find the equation.

Given points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$, and $P_3 = (x_3, y_3, z_3)$. First, what we can do is find a vector that is perpendicular to the plane that we are trying to find. We can find the vector that goes from P_1 to P_2, $\overrightarrow{P_1P_2}$ and the vector that goes from P_1 to P_3, $\overrightarrow{P_1P_3}$. These vectors can also be expressed as $\vec{P_2} - \vec{P_1}$ and $\vec{P_3} - \vec{P_1}$, respectively. If this is not clear, prove it! Therefore, we get:

$$\overrightarrow{P_1P_2} = \vec{P_2} - \vec{P_1} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$
$$\overrightarrow{P_1P_3} = \vec{P_3} - \vec{P_1} = (x_3 - x_1)\hat{\imath} + (y_3 - y_1)\hat{\jmath} + (z_3 - z_1)\hat{k}$$

Then, we can find a vector perpendicular, or normal, to the plane, $\vec{N}$ by taking $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$. Notice that the order in which the cross product is taken does not matter, because both vectors will be normal to the plane.

$$\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}\hat{\imath} - \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}\hat{k}$$

$$\begin{aligned}\vec{N} = [(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]\hat{\imath} - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)]\hat{\jmath}\\ + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\hat{k}\end{aligned}$$

Well, that's bulky! But don't worry, when we're working with real numbers it is much cleaner. Recall that when any two vectors are orthogonal, their dot product is zero. Because of this, if we take the dot product of any vector that lies on the plane and $\vec{N}$, it will be zero. Given an arbitrary point on the plane $P = (x, y, z)$ and the point P_1, we know that

$$\vec{N} \cdot \overrightarrow{PP_1} = \vec{N} \cdot (\vec{P_1} - \vec{P}) = 0$$

Then, because of the distributive property of dot products,

$$\vec{N} \cdot \vec{P_1} - \vec{N} \cdot \vec{P} = 0$$

$$\vec{N} \cdot \vec{P_1} = \vec{N} \cdot \vec{P}$$

Writing this all out,

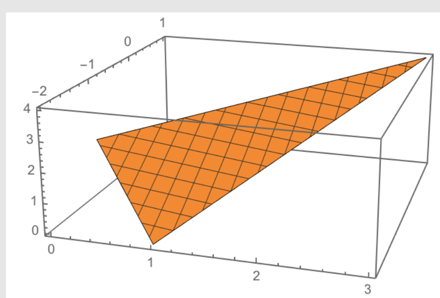
$$\begin{aligned}
&[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)](x_1) - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)](y_1) \\
&\quad + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)](z_1) = [(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)](x) \\
&\qquad - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)](y) + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)](z)
\end{aligned}$$

This is the equation for the plane. Obviously, memorizing this formula would be incredibly hard, and probably wouldn't earn you full credit on a test. Because of this, a comprehensive understanding of the method we are using is necessary. Let's look at an example.

Example 3.4

Find the equation of the plane that goes through the points $P_1 = (1, -2, 0)$, $P_2 = (3, 1, 4)$, and $P_3 = (0, -1, 2)$.

At a glance, the plane, when graphed, would look like:



Obviously, the plane isn't restricted to this single triangle - it goes on infinitely. This representation just helps us get a better understanding of the properties of it.

Lets start off by finding the normal vector, $\vec{N}$:

$$\overrightarrow{P_1P_2} = 2\hat{i} + 3\hat{j} + 4\hat{k} \qquad \overrightarrow{P_1P_3} = -1\hat{i} + 1\hat{j} + 2\hat{k}$$

$$\begin{aligned}\vec{N} &= \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \\ &= \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \hat{k} \\ &= 2\hat{i} - 8\hat{j} + 5\hat{k}\end{aligned}$$

Letting point $P = (x,y,z)$ on the plane, we know that $\vec{N} \cdot \overrightarrow{PP_1} = 0$

$$\vec{N} \cdot (\vec{P_1} - \vec{P}) = 0$$

$$\vec{N} \cdot \vec{P_1} = \vec{N} \cdot \vec{P}$$

$$(2\hat{i} - 8\hat{j} + 5\hat{k}) \cdot (1\hat{i} - 2\hat{j} + 0\hat{k}) = (2\hat{i} - 8\hat{j} + 5\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

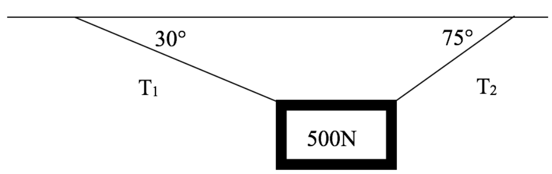
$$2 + 16 + 0 = 2x - 8y + 5z$$

We now have our final equation for the plane:

$$\boxed{2x - 8y + 5z = 18}$$

4.9 Problems

1. For the following vectors, find a) the unit vector of both $\vec{A}$ and $\vec{B}$, b) $\text{proj}_{\vec{B}}\vec{A}$, c) $\text{proj}_{\vec{A}}\vec{B}$, and d) the cosine between the two angles.
   (a) $\vec{A} = (2,-1)$; $\vec{B} = (-1,1)$
   (b) $\vec{A} = (3,5)$; $\vec{B} = (2,-3)$
   (c) $\vec{A} = (-1,-1,3)$; $\vec{B} = (3,3,-2)$
   (d) $\vec{A} = (\pi,3,-1)$; $\vec{B} = (2\pi,-3,7)$
2. Find the angles of the triangle whose vertices are
   (a) $(2,-2,-1),(4,-3,2),(-1,2,-5)$
   (b) $(2,-1,1),(1,-3,-5),(3,-4,-4)$
3. If $\vec{a} = -8\beta\hat{i} + 15\beta\hat{j} - \beta\hat{k}$ and $\vec{b} = 6\beta\hat{i} - 2\beta\hat{j} + 7\beta\hat{k}$, where β is a constant, find:
   (a) The unit vector in the opposite direction of $\vec{a}$
   (b) The parallel component of $\vec{a}$ to $\vec{b}$
   (c) The perpendicular component of $\vec{b}$ to $\vec{a}$
4. Find the equation of the plane that passes through the points $(1,-2,0)$, $(3,1,4)$, and $(0,-1,2)$
5. There is a block that weighs 500 Newtons (a measure of weight, which is a force) suspended from two wires, as pictured. If the box is at rest, find the magnitudes of the forces T_1 and T_2.



6. A plane of weight W is flying horizontally at speed S_p directly north. There is a wind of speed S_w blowing at angle θ east of north.
   (a) At direction must the plane "aim" in order for the resulting direction to be directly north? Answer in terms of S_p, S_w, and θ.
   (b) Now, consider the vertical forces on the plane. If the engine can provide a vertical force of magnitude F_e on the plane, then what vertical angle must the plane "aim" at in order to be traveling horizontally? Answer in terms of W and F_e.

7. Find the volume of the parallelepiped bound by the vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + 8\hat{k}$, $\vec{b} = 5\hat{\imath} + 9\hat{\jmath} - 4\hat{k}$, and $\vec{c} = -2\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$.

8. Three forces are pushing a box, but the box is at rest. Two forces are known: $\vec{F}_1 = [115N, 35°]$ and $\vec{F}_2 = [300N, 330°]$. What is $\vec{F}_3$?

9. Consider an arbitrary vector, $\vec{A}$, and a unit vector $\hat{n}$ in a different direction than $\vec{A}$. Show that $\vec{A} = (\vec{A} \bullet \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$

10. Three rodeo clowns lasso a bull to try to pull it out of the ring. Clown A pulls with 125N of force at an angle of 191°. Clown B pulls 100N of force at an angle greater than the angle at which Clown A pulls, and Clown C pulls with a force of 140N at an angle less than the angle at which Clown A pulls. The smaller angle between Clown B's and Clown C's lasso is 76°. If the bull is resisting with a force of 315.059 N at an angle of 15.581° to remain still, calculate the angles at which Clown B and Clown C are pulling.

11. Consider the following points:

$$P : (2, -1, 5) \quad A : (-1, 3, 2) \quad B : (1, -2, 4) \quad C : (3, 0, -1)$$

(a) Find the equation of the plane formed by $A, B,$ and C

(b) Find the distance from point P to the plane (we define this as the distance from P to the point on the plane **closest** to P).

12. Find the distance between the following planes:

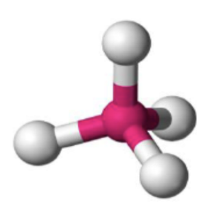
$$3x + 2y - 2z = 5$$
$$3x + 2y - 2z = 12$$

13. Find the distance between the point $\mathscr{P} = (2, -1, 5)$ and the plane going through the points $(-1, 3, 2), (1, 2, 4),$ and $(3, 0, -1)$.

14. Find a unit vector that is parallel to the line formed by the intersection of the following planes:

$$3x + 2y - 2z = 5$$
$$-x + 3y + 5z = 10$$

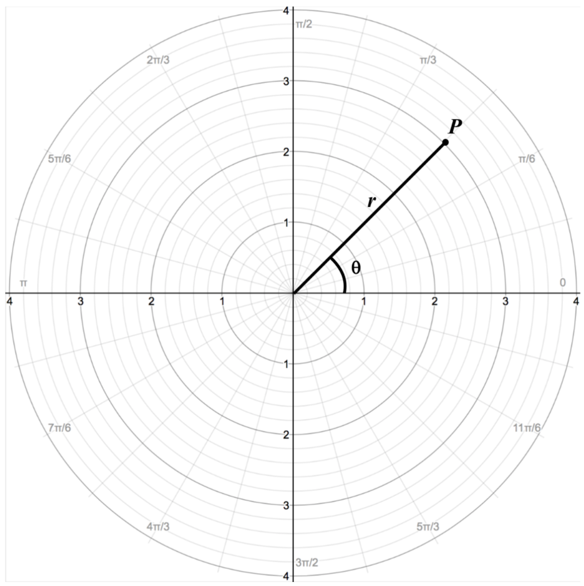
15. Consider the following structural diagram of methane, CH_4:



Given that the hydrogen atoms are represented by a white sphere and the carbon pink, determine the bond angle of a H − C − H bond.

5 Polar Coordinates

Polar coordinates are a completely new way of representing a point. In the polar plane, each point is represented not as a horizontal and vertical component, but as a magnitude (or radius) and direction. We put polar coordinates in the form of $[r, \theta]$, where r is the distance from the origin (also known as the radius) and θ is the angle made from the horizontal axis, as shown in the figure below:



Point P can represented by the polar coordinates $[3, \frac{\pi}{4}]$. If we wanted to convert P into rectangular coordinates, we simply would separate the radius into its horizontal and vertical components:

$$[r, \theta] = (r\cos\theta, r\sin\theta) \tag{45}$$

Likewise, an ordered pair (x, y) can be converted into polar coordinates:

$$(x, y) = [\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}] \tag{46}$$

If this formula is confusing, prove it yourself! Essentially, though, the use of the inverse tangent function is due to the fact that x, y, and r can be represented in a right triangle, with r the hypotenuse, y the side opposite of θ, and x the adjacent side. Because of this, we know that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$. Thus, $\theta = \tan^{-1}\frac{y}{x}$.

Example 5.1

Find the polar coordinates of the rectangular point $(3, -5)$

First, we can find r:

$$\begin{aligned} r &= \sqrt{(3)^2 + (-5)^2} \\ &= \sqrt{34} \end{aligned}$$

Now, we can find θ:

$$\begin{aligned} \theta &= \tan^{-1} -\frac{5}{3} \\ &= -59° \end{aligned}$$

Thus, our polar coordinate then becomes

$$\boxed{[\sqrt{34}, -59°]}$$

There are several properties of polar coordinates that give us more versatility when operating with them. First, if a coordinate has a negative radius, say $[-r, \theta]$, then that is just the same as $[r, \theta + \pi]$, or a 180° rotation of the original coordinate. This makes sense - a negative radius means you go "backward", whereas a positive radius means you go "forward". But, backward the θ direction is forward in the $\theta + \pi$ direction. Therefore, we can say

$$[-r, \theta] = [r, \theta + \pi] \tag{47}$$

In addition, there are a couple more polar trigonometry identities we can derive from the right triangle setup described above. Using the same right triangle logic that was used to get $\tan\theta = \frac{y}{x}$, we can get

$$r^2 = x^2 + y^2 \tag{48}$$

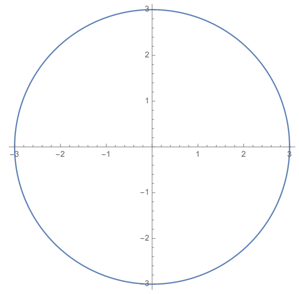
$$\cos\theta = \frac{x}{r} \tag{49}$$

$$\sin\theta = \frac{y}{r} \tag{50}$$

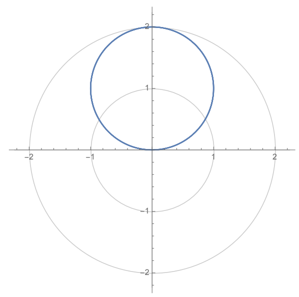
Using these identities, we can manipulate rectangular equations in a number of ways to get the polar equation, or vice versa.

5.1 Circles

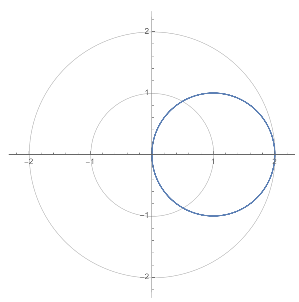
Similar to rectangular equations, which usually compute the y-value as a function of x, polar equations compute the radius as a function of θ, $r(\theta)$. The first type of polar equation we will go over are circles. Circles are one of the more simple polar equations we will encounter. Although a circle formula in the rectangular plane may be complicated, it is much easier in the polar plane. The general polar formula for a circle is $r(\theta) = A$, $r(\theta) = A\sin\theta$, or $r(\theta) = A\cos\theta$. The first polar form results in a circle centered around the pole, while the second and third forms result in a circle centered around one of the axes with a center $\frac{A}{2}$ distance from the pole. Lets start off by taking a look at the simplest form: $r(\theta) = 3$



Pretty simple, right? The radius is constant, which is actually the definition of a circle: a collection of all the points equidistant from one point. Now, lets complicate things with the introduction of θ into the function. Similar to x in a rectangular equation, θ is a variable that causes the overall radius to be non-constant. First, lets start with $r(\theta) = 2\sin\theta$:



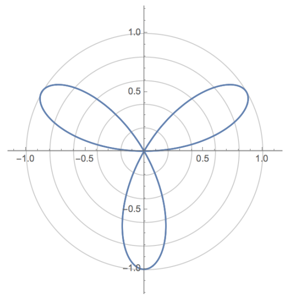
Let's look at a couple points on this graph. The center is at $[\frac{\pi}{2}, 1]$, which fits the statement above, as the center is always $\frac{A}{2}$ units away from the pole. In addition, when $\theta = 0$, $r = 0$. This makes sense; $2\sin(0) = 0$. Now, lets look at $r(\frac{\pi}{2})$: this is 2, which also makes sense. How about $r(\frac{7\pi}{6})$? This would be -1, which means the r would be negative and, as a result, go in the $\frac{\pi}{6}$ direction. However, this is just the same as $r(\frac{\pi}{6})$, due to the properties of the sine function. Something interesting is going on here; the function is actually overlapping itself once θ gets to π. Similarly, the cosine graph exhibits similar features; consider the graph $r(\theta) = 2\cos\theta$:



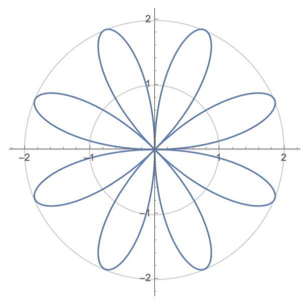
With a radius of 2 and a center at [0, 1], this graph also holds true to the idea that the center is $\frac{A}{2}$ units away from the pole. Similar to the cosine graph above, this equation also "overlaps" itself over a period of 2π. As it turns out, the graph of $r(\theta)$ over the domain of π is just the same as the graph over the domain of π. The same idea applies to the graph of a sine circle as well.

5.2 Rose Curves

Rose curves are a type of polar equation that take the general form $r(\theta) = A\sin(B\theta)$ or $r(\theta) = A\cos(B\theta)$. The coefficient B results in the graph having "petals". This is because as B increases, the frequency, or number of periods over a given interval, increases. Therefore, as B goes up, the number of petals goes up [3]. Another way of thinking about circles is as a rose curve with just one petal (or a B value of 1). Consider the graph $r(\theta) = \sin(3\theta)$ below:



Since our A value is 1, the length of the petals is 1. In addition, because our B is 3, we have 3 petals. We encounter a problem, though, when B is even. Consider the graph of $r(\theta) = 2\sin(4\theta)$:



There are... 8 petals? As it turns out, when B is even, the number of petals is $2B$. Similar to the circle situation described previously (when B is odd), the graph retraces itself after θ gets past π. Take the equation $r(\theta) = \sin(3\theta)$ for example. To start off, we can compare the differences in radii from θ and $\theta + \pi$. First, by plugging in $\theta + \pi$:

$$\begin{aligned} r(\theta+\pi) &= \sin[3(\theta+\pi)] \\ &= \sin(3\theta+3\pi) \\ &= \sin(3\theta+\pi) \\ &= -\sin(3\theta) \end{aligned}$$

Although this radius is negative, the $\theta + \pi$ is in the opposite direction from θ. This results in the graph retracing itself; although $r(\theta) = -r(\theta + \pi)$, the directions are opposite, so the positive and negative radius end up in the same place. This general idea applies to **any** function with an odd value of B. The angle inside the sine/cosine function will always be $\theta + B\pi$, and if B is odd, then

[3] This is not always true, however, when we go from an even B to an odd B

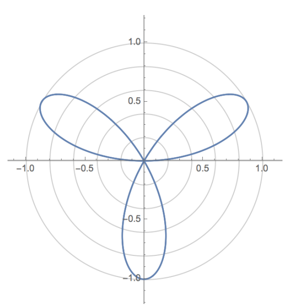
the angle is just $\theta+\pi$. This same logic applies to the cosine function.
Now, let's look at the case when our B value is even. Consider the function $r(\theta)=\cos(4\theta)$. If we compare θ and $\theta+\pi$ like we did before, we get:

$$\begin{aligned} r(\theta+\pi) &= \cos[4(\theta+\pi)] \\ &= \cos(4\theta+4\pi) \\ &= \cos(4\theta) \end{aligned}$$

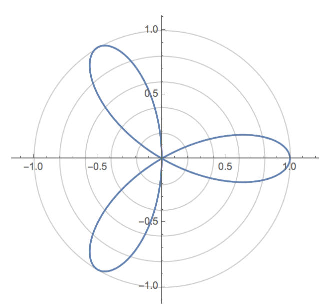
Although $r(\theta)=r(\theta+\pi)$, the directions are opposite; this results in no overlap of petals, which means there are **twice** as many petals as the value of B.

But how, exactly, does B dictate how many petals there are? There are many ways to think about this, but we choose to think about petals by the number of times $r(\theta)$ intersects the pole. Considering the equation $r(\theta)=\cos(3\theta)$, we can find the number of times it crosses the pole on the domain $[0,2\pi)$. Although each petal technically goes through the pole twice, each contact point of a petal is the same contact point as another petal; this results in the number of times $r(\theta)$ goes through the pole to be the number of petals it has. Setting $\cos(3\theta)=0$, we find that $\theta=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2}$, and $\frac{11\pi}{6}$. However, since $r(\theta)=r(\theta+\pi)$ when B is odd, we can negate any values above $\theta=\pi$. This results in 3 remaining values of θ, which makes sense; there are three petals on this rose curve. The same logic applies when B is even, except all values of θ from 0 to 2π must be considered. Although the number of zeroes on a certain domain is independent of B being even or odd, it has twice as many zeroes because its domain is twice as large.

Although subtle, the trigonometric operator does have an effect on the graph, as shown in the graphs of $r(\theta)=\sin(3\theta)$ and $r(\theta)=\cos(3\theta)$ below:



$r(\theta)=\sin(3\theta)$

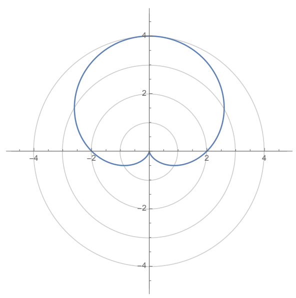


$r(\theta)=\cos(3\theta)$

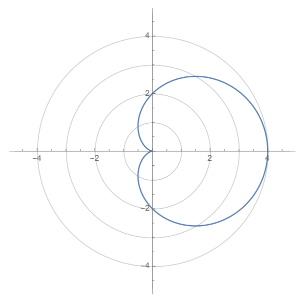
As the graphs show, when r=0, $\sin(3\theta)=0$, but $\cos(3\theta)=1$. This makes sense, as that is the behavior of a normal sine and cosine function. This same property applies throughout any value of θ; the graphs are the same except for a rotation of $\frac{\pi}{6}$. This makes sense as well. Normally, the shift between a sine and cosine graph would be $\frac{\pi}{2}$. But since we have a B value of 3, we multiply that shift by $\frac{1}{3}$ - if this is confusing, skip back to the first unit on trigonometric functions.

5.3 Limaçons

Limaçons are another type of polar equation that take on the general form $r(\theta) = A + B\sin\theta$ or $r(\theta) = A + B\cos\theta$, where A and B can be any real number. Limaçons are arguably the most difficult polar equation to graph, as they have many minute details in the equation that effect the overall graph. To start off, the graph of a Limaçon equation that has a cosine function has an obvious difference from that of a sine function; the symmetry for the sine Limaçon is along the $\frac{\pi}{2}$ axis, while the cosine Limaçon has symmetry about the polar axis, or the right horizontal axis. Consider the graphs $r(\theta) = 2 + 2\sin\theta$ and $r(\theta) = 2 + 2\cos\theta$:

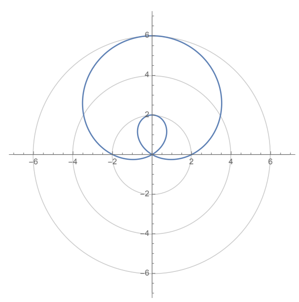


$r(\theta) = 2 + 2\sin(\theta)$



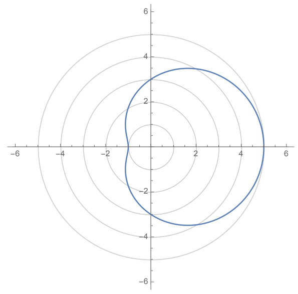
$r(\theta) = 2 + 2\cos(\theta)$

Now, what if A is not equal to B? Depending on the specific values of A and B, the Limaçon may have an inner loop. Take a look at the graph of $r(\theta) = 2 + 4\sin(\theta)$:

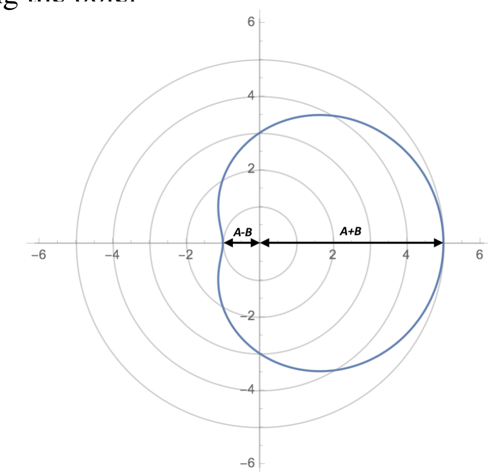


As it turns out, when A is less than B, the graph has an inner loop. This is because for certain values of θ, the r becomes negative, thus forming an inner loop. In this specific equation, once θ passes $\frac{7\pi}{6}$, the radius becomes negative; this can be tested in the equation as well; $r(\frac{7\pi}{6}) = 0$, and from then on r is negative. Once r is negative, the r does not go forward, but backward.

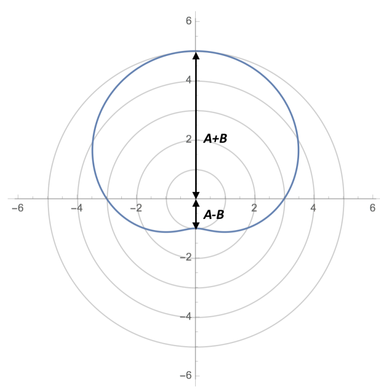
Now, what if $A > B$? Well, as you might expect, the graph has an almost-circular shape. Consider the graph of $r(\theta) = 3 + 2\cos(\theta)$:



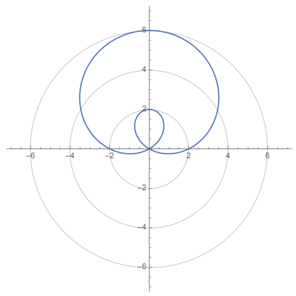
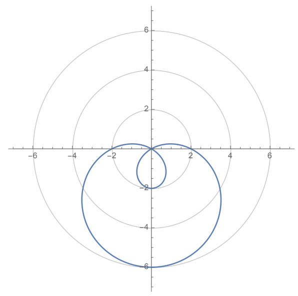
The graph almost touches the pole. However, there is never a negative r value, so there is no inner loop - this can be proven algebraically as well. If we were to graph $f(x) = 3+2\cos x$, it would never touch the x-axis. This is analogous to the polar graph touching the pole.



By now, you may be noticing a pattern among the graphs. A and B actually give us more than a general shape; they tell us two explicit coordinates of the graph:
The smaller distance on the axis of symmetry from the pole to the curve is $A-B$, while the larger distance is $A+B$. This also known as the minimum radius and maximum radius, respectively. This makes sense, because when $\theta = 0$, $B\cos\theta = B$. Therefore, r is just $A+B$. However, when $\theta = \pi$, $B\cos\theta = -B$, making r $A-B$. This same concept applies to Limaçons with inner loops as well; however, $A-B$ is negative, which simply means a radius in the opposite direction. With sine Limaçons, the same rule applies along the axis of symmetry; the minimum radius is $A-B$, while the maximum radius is $A+B$:

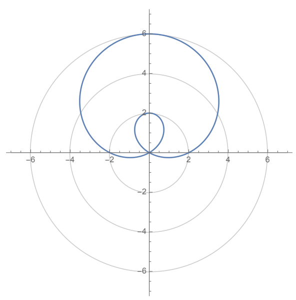
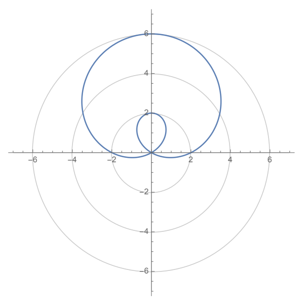


Now, what if A or B is negative? As you may expect, when B is negative, the graph is reflected along the axis perpendicular to the axis of symmetry. For example, compare the graphs of $r(\theta) = 2+4\sin\theta$ and $r(\theta) = 2-4\sin\theta$:

$r(\theta) = 2 + 4\sin\theta$ $\qquad$ $r(\theta) = 2 - 4\sin\theta$

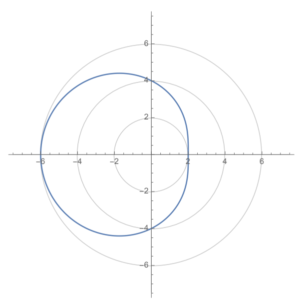
Now, what about if A is negative? The result is surprising:

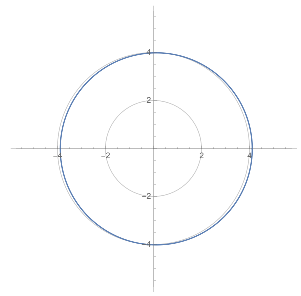
$r(\theta) = 2 + 4\sin\theta$ $\qquad$ $r(\theta) = -2 + 4\sin\theta$

This property is beautiful; essentially, every r value is switched, but in a manner that contributes nothing to the overall shape of the graph. How is it that $r(\frac{\pi}{2})$ have different values in each function, yet they look exactly the same when graphed? When A is negative, the negative radii are the largest in size, while when A is positive, the positive radii are the largest.

In addition, Limaçons can take on another form, which is convex. This only happens if A is at least twice as large as B. Consider the graph $r(\theta) = 4 - 2\cos\theta$:



At and around $r(0)$, the graph is flat. In fact, the curvature at *r(0)*, for reasons you will explore in calculus, is zero when $A = 2B$. This results in the Limaçon having no inward curve. Because of this, the bigger $|\frac{a}{b}|$ is, the more circular the graph is:



$r(\theta) = 4 + 0.1\cos\theta$

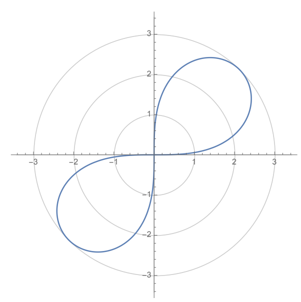
This is because as B gets smaller, the term $B\cos\theta$ has less of an effect on the overall function. Because of this, A becomes the dominating factor in the function, which results in a near-circle due to the almost-constant radius.

5.4 Lemniscates

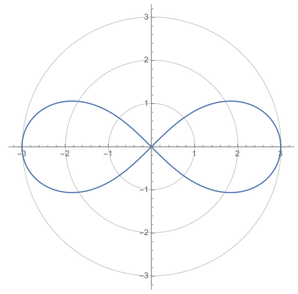
Lemniscates can be thought of as 2-petal rose curves. As you've probably figured out by now, it is impossible to have a 2-petal rose using the general formula for a rose curve $r(\theta) = A + B\sin\theta$ (or cosine). If this is unclear, review the section on rose curves. The general form for a lemniscate is

$$r^2 = A^2\sin(2\theta) \qquad \text{or} \qquad r^2 = A^2\cos(2\theta)$$

Notice that these equations have only one constant; B. This is because the number of petals is given. Let's take a look at $r^2 = 9\sin(2\theta)$:



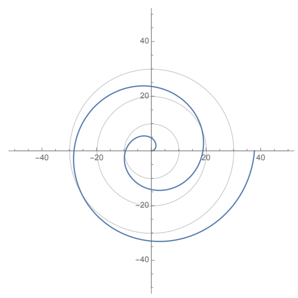
As we can see, the length of the rose petals is 3, or $\sqrt{A}$. This makes sense; if we take the square root of both sides, we can just graph two equations; $r(\theta) = 3\sqrt{\sin(2\theta)}$ and $r(\theta) = -3\sqrt{\sin(2\theta)}$. In addition, the sine leminscate follows the same general rule as the rose petal regarding the orientation of the petals; because $\sin(0) = 0$ and $\sin[(2)(\frac{\pi}{2})] = 0$, the petal is longest at $\frac{\pi}{4}$, because of the 2θ in the sine (which maximises $\sin(2\theta)$ at $\frac{\pi}{4}$). When the lemniscate is in the form $r^2 = A^2\sin(2\theta)$ it is important to note that the petal length is A, rather than A^2. In addition, we can see that the axis of symmetry is on the line $\theta = \frac{\pi}{4}$ - in the polar plane, this forms a line.
Similar to the sine lemniscate, the cosine lemniscate has the same general shape with a different orientation; consider the graph of $r^2 = 9\cos(2\theta)$



The different orientation makes sense; $r(0) = \sqrt{9\cos 0} = 3$, while when the sine function is involved, it is zero.
Overall, while lemniscates have one of the more complicated polar formulas, they are one of the easier graphs to draw. The only features to take into account are the radius and orientation.

5.5 Spirals

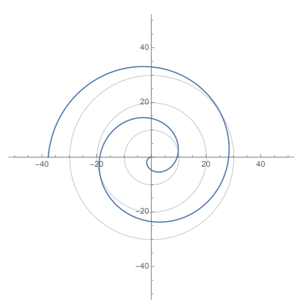
Finally, the last polar graph we will cover is the spiral. Spirals have the general form $r = A\theta$. Spirals are one of the few shapes that do not have a trigonometric operator in it. Rather, the radius is directly proportional to the angle, θ. Consider the graph of $r = 3\theta$:



As shown in the graph, as θ increases, the r value increases at a constant rate.
One important thing that A gives us is the way the spiral grows; more specifically, if A is negative, then the spiral is essentially shifted an angle of π around the origin. Mathematically, this statement means:

$$r(\theta) = -A\theta = A(\theta + \pi) \tag{51}$$

However, one thing still remains the same among all spiral graphs; a spiral will grow in a counter-clockwise direction regardless of the value of A. Consider the graph of $r(\theta) = -3\theta$:



Clearly, the graph is just rotated by an angle of π. This is because every radius is negative due to the value $-A$, which results in every radius being rotated by an angle of π.

5.6 DeMoivre's Theorem

DeMoivre's Theorem allows us to represent a complex number to a power, $(a+bi)^n$, in terms of sine and cosine. Before we delve into any actual math, there is an abbreviation that will appear numerous times throughout this section:

$$\text{cis}\,\theta = \sin\theta + i\cos\theta \tag{52}$$

In order to find the relationship between these two seemingly-unsimilar aspects of math, we must use a proof method called mathematical induction. See page 154 if you are unfamiliar with mathematical induction.

As it turns out, we can use induction to prove DeMoivre's theorem (see page 154). In this case, we are trying to prove the following:

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n\,\text{cis}(n\theta)$$

$$\text{Where } r = \sqrt{a^2+b^2}$$

We start off by proving the identity is true for $n = 0$:

$$(a+bi)^0 = r^0[\cos(0\theta) + i\sin(0\theta)]$$

$$1 = 1(1+0)$$

$$1 = 1$$

Now, we assume the statement is true for $n = k$. Before we do this, however, it is important to understand how we can separate $(a+bi)^k$; consider the magnitude of this complex point, r, which is the distance from the point to the origin. We can take out r from $(a+bi)$ to get $r(\cos\theta + i\sin\theta)$, which is simply r multiplied by the horizontal and vertical components, respectively, of the complex point. Namely, $a+bi = r(\cos\theta + \sin\theta)$ We use this information in the boxed equation.

$$(a+bi)^k = r^k[\cos(k\theta) + i\sin(k\theta)]$$

$$\boxed{[r(\cos\theta + i\sin\theta)]^k = r^k[\cos(k\theta) + i\sin(k\theta)]}$$

$$r^k(\cos\theta + i\sin\theta)^k = r^k[\cos(k\theta) + i\sin(k\theta)]$$

Then, we prove for $n = k+1$. For this part of the proof, we will manipulate the left side only:

$$(a+bi)^{k+1} = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

$$[(r)(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

$$(r)^{k+1}(\cos\theta + i\sin\theta)^{k+1} = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$
$$r^{k+1}(\cos\theta + i\sin\theta)^{k}(\cos\theta + i\sin\theta) = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

At this point, we must recognize that we already know what $(\cos\theta + i\sin\theta)^k$ is - we assumed it was equal to $\cos(k\theta) + i\sin(k\theta)$, as shown above. Because of this, we can substitute.

$$r^{k+1}[\cos(k\theta) + i\sin(k\theta)](\cos\theta + i\sin\theta) = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

Now, we simply have to brute force it.

$$r^{k+1}[\cos(k\theta)\cos\theta + \cos(k\theta)i\sin(k\theta) + i\sin(k\theta)\cos\theta - \sin(k\theta)\sin\theta]$$
$$= r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

Using double angle identities,

$$r^{k+1}[\cos(k\theta+\theta) + i\sin(k\theta+\theta) = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$
$$r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]] = r^{k+1}[\cos[(k+1)\theta] + i\sin[(k+1)\theta]]$$

Although the proof was bulky, we have proved a very important identity:

$$(a+bi)^n = r^n \operatorname{cis}(n\theta) \tag{53}$$

Example 5.2

Write $(\sqrt{3}+i)^7$ in $a+bi$ form.

First, we can find r:

$$\begin{aligned} r &= \sqrt{(\sqrt{3})^2 + 1^2} \\ &= 2 \end{aligned}$$

Now, we know that $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$; draw a right triangle and solve if this is unclear. Because of this, $\theta = \frac{\pi}{6}$. Therefore,

$$\begin{aligned} (\sqrt{3}+i)^7 &= \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^7 \\ &= 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \\ &= 128\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= -64\sqrt{3} - 64i \end{aligned}$$

$$\boxed{(\sqrt{3}+i)^7 = -64\sqrt{3} - 64i}$$

DeMoivre's Roots Theorem

Since we have proven this for any value of n greater than 0, this identity applies to any n'th root as well, as any root of a positive number is greater than zero. Namely,

$$\sqrt[n]{a+bi} = \sqrt[n]{r}\operatorname{cis}\frac{\theta+2k\pi}{n}$$

Where $k \in 1,2,3...n-1$. This because the frequency increases when n is in the denominator, which means we have more roots. The identity above is valid because it is just the same as DeMoivre's Theorem but with n in the denominator of the exponent - made possible by the exponential property $\sqrt[n]{a+bi} = (a+bi)^{\frac{1}{n}}$.
Thus, we have DeMoivre's Roots Theorem:

$$(a+bi)^{\frac{1}{n}} = r^{\frac{1}{n}}\operatorname{cis}\frac{\theta+2k\pi}{n} \tag{54}$$

5.7 Problems

1. Graph the following equations:

   (a) $r(\theta) = 3 - 2\sin\theta$

   (b) $r(\theta) = 4\sin(4\theta)$

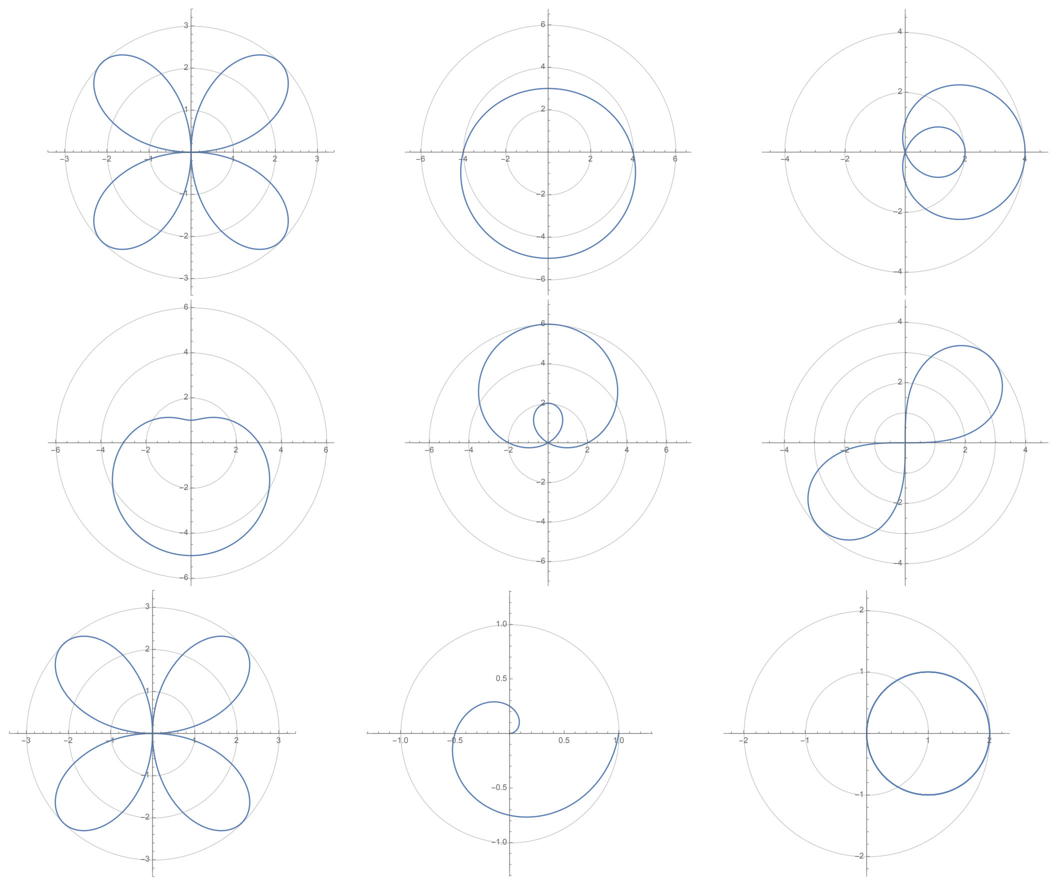
   (c) $r(\theta)^2 = 9\sin(2\theta)$

   (d) $r(\theta) = 3 - 3\cos\theta$

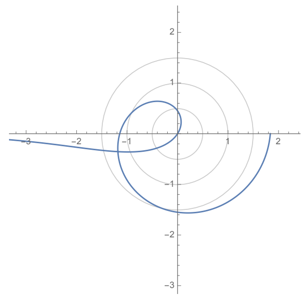
   (e) $r(\theta) = 4\cos 3\theta$

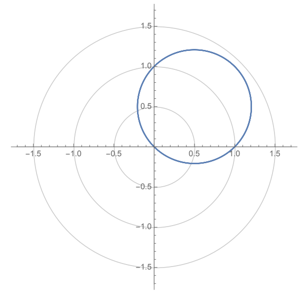
   (f) $r(\theta) = \frac{1}{\theta}$

2. Write the equations for each of the following graphs:



3. ★ Write the equations for each of the following graphs:





4. Graph the following equations and find any points of intersection:

$$r(\theta) = 4\sin(2\theta) \qquad r(\theta)^2 = 2\cos(2\theta)$$

5. There is a ferris wheel at an amusement park. The shape of the ferris wheel can be modeled by the polar equation $r = 20\cos\theta$, where r is in feet. If one full revolution of the ferris wheel takes 35 seconds, how many feet off the ground is the rider after 23 seconds? How much time has passed when the rider is 17 feet off the ground?

6. Convert each of the following rectangular equations into polar equations:

   (a) $x^2 + y^2 = 3$
   (b) $x^2 + 4y^2 = 1$
   (c) $[(x-3)^2 + y^2][(x+3)^2 + y^2] = 81$
   (d) $(x^2 + y^2 + 10x)^2 = 16(x^2 + y^2)$

7. Convert each of the following polar equations into rectangular equations:

   (a) $r(\theta) = \frac{9}{8\cos\theta - 10\sin\theta}$
   (b) $r(\theta) = \frac{\sec\theta}{\tan\theta}$
   (c) $r(\theta) = \frac{2}{\sin\theta - 3\cos\theta}$
   (d) $r(\theta) = \sqrt{r\cos\theta + 2}$
   (e) $r(\theta) = \sin(2\theta)$

8. Two polar points are plotted on a polar plane: $P_1[6, \frac{\pi}{4}]$ and $P_2[3, \theta]$. If the distance between P_1 and P_2 is 6, find θ.

9. State whether or not the following equations, when graphed, will be a parabola.

   (a) $r(\theta) = \frac{1-\cos^2\theta}{\sec\theta}$
   (b) $r(\theta) = \frac{\sin\theta}{1-\sin^2\theta}$
   (c) $r(\theta) = \cot\theta\sec\theta$

(d) $r(\theta) = \frac{\cot^2\theta + 1}{\sec\theta}$

(e) $r(\theta) = \ln\theta$

10. Solve $-16\sqrt{3} + 16i - x^5 = 0$. Leave your answers in polar form.

11. Write the following in standard form, $a + bi$:

    (a) $[\sqrt{2}(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8})]^6$

    (b) $(\sqrt{2} + i\sqrt{2})^4$

    (c) $(\sqrt{3} + i)^7$

    (d) $(\sqrt{5} + i\sqrt{5})^4$

    (e) $(2 + i\sqrt{7})^5$

    (f) $\sqrt[5]{\sqrt{3} + i}$

    (g) $\sqrt[4]{-1 + i\sqrt{3}}$

    (h) $(-81i)^{\frac{1}{4}}$

12. There is an amusement park ride named the Typhoon. The ride is similar to a ferris wheel; however, the shape is not circular. The shape can be modeled by the equation $r(\theta) = 28\sin(7\theta)$, where r is in meters. The Typhoon takes 70 seconds to make one complete revolution and the rider starts at the center of the ride at $t = 0$ and $\theta = 0$. Assume that as t increases, θ increases.

    (a) Where is the rider after after 20 seconds?

    (b) The rider's mom wants to give him a high-five. Her coordinates relative to the center of the Typhoon are $[30, \frac{\pi}{3}]$. When will the rider be closest to his mom? Realistically, could they high-five?

    (c) The rider's sister throws a ball to him from the rectangular coordinate $(50, 75)$. The ball travels at a horizontal speed of 10 m/s (ignore acceleration due to gravity). If the sister throws the ball as soon as the ride starts, where should she aim relative to herself?

6 Functions

This chapter will go over the basic properties of functions. There is no particular order to the concepts explained, as the purpose of this chapter is to provide a basic knowledge that we will delve much deeper into in the next several chapters. To start off, we define a **function** as a relation in which each x value is associated with exactly one y value. Because of this, in a function, any given x value cannot have multiple y values. A **relation** is a relationship between sets of values. In math, this *usually* means a relationship between x and y values. The main distinction between a relation and a function is that a relation can have multiple y-values associated with one x-value, while a function cannot. Try these exercises below - state whether these are functions, or simply relations. The best way to approach problems similar to these is to graph first (or picture them in your head, if you have a good idea of what they look like), and then use the vertical line test.

1) $y = 2x - 1$ 2) $y = x^2 - 4x + 2$ 3) $x^2 - y^2 = 4$ 4) $f(x) = x^5 - \frac{1}{3}x^4 + 2x^3 - 3$

5) $xy = 3$ 6) $f(x) = x\cos x$ 7) $y = e^x$ 8) $y = \sqrt{\ln(x^4) - y}$

An **odd function** is a function that is symmetrical about the origin. Namely, $f(x) = -f(-x)$. An example of this is $f(x) = \sin x$. An **even function** is a function that is symmetrical about the y-axis. Thus, $f(x) = f(-x)$. The function $f(x) = \cos(x)$ illustrates this. Determine whether the following functions are even, odd, or neither:

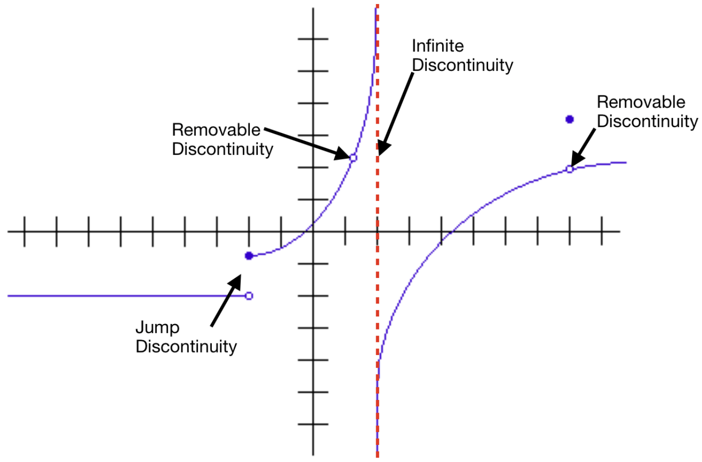
1) $f(x) = -\cos x$ 2) $y = x^3$ 3) $f(x) = x^2 - \cos x$ 4) $f(x) = x^2 - \sin x$

6.1 Continuity of Functions

Functions can be continuous or discontinuous as well. For a function to be continuous at a given point, the following conditions must be satisfied, where c the x-value which we are testing for continuity in $f(x)$:

- $f(c)$ exists
- $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$
- $\lim_{x\to c} f(x) = f(c)$

These conditions may seem hard to understand at first. To elucidate, we will start with the first condition. All that says is that if you have any given x-value, c, (whether that be 1, 7, 284, or any other real number you can imagine), then $f(c)$ *exists*. Without this condition satisfied, the resulting graph would have a **removable discontinuity** (also known as a hole). The second condition states that the **limit** as x approaches c from the negative (or left-hand) side of $f(x)$ is the same value as the limit as x approaches c from the positive (or right-hand) side. Without this condition satisfied, we would have a **jump discontinuity**. Finally, the third condition states that the limit as x approaches c of $f(x)$ is the same value as $f(c)$. This condition could result in an **infinite discontinuity**.



Determine if the following functions are continuous at their given c-values. If not, state what type of discontinuity it is [4].

1) $f(x) = x^2 - 5, c = 3$ 2)$f(x) = \frac{x}{x}, c = 0$ 3) $f(x) = \frac{x+2}{x^2-4}, c = 2$

6.2 End Behavior

The end behavior is what happens to the graph as x approaches infinity (or negative infinity). Mathematically, we can define left end behavior as

$$\lim_{x\to\infty^-} f(x)$$

and the right end behavior as

$$\lim_{x\to\infty} f(x)$$

All these limit statements mean is "the value $f(x)$ approaches as x gets infinitely positive or negative". For example, if we wanted to find the left end behavior of the function $f(x) = \frac{3x^2-2x+5}{x^3}$, we can set up a limit and solve algebraically:

$$\lim_{x\to\infty}\left(\frac{3x^2-2x+5}{x^2}\right) = \lim_{x\to\infty}\left(\frac{3x^2}{x^2} - \frac{2x}{x^2} + \frac{5}{x^2}\right)$$

$$= \lim_{x\to\infty}\left(\frac{3x^2}{x^2}\right) - \lim_{x\to\infty}\left(\frac{2x}{x^2}\right) + \lim_{x\to\infty}\left(\frac{5}{x^2}\right)$$

Now, there is something important to notice here. On the middle limit, we have an x term in the numerator and an x^2 term in the denominator. As x gets larger and larger, the denominator gets much larger than the numerator (plug in $x = 100$ and test it out for yourself). Since the denominator will get an infinite amount larger as x goes to infinity, the limit is zero. Namely,

$$\lim_{x\to\infty}\left(\frac{2x}{x^2}\right) = 0$$

[4] 1) Continuous, 2) Discontinuous; removable, 3)Discontinuous; infinite

The same logic applies for the right limit; there is simply a 5 in the numerator and an x^2 in the denominator. Because 5 will always stay the same regardless of the value of x but the x^2 will increase as x increases, the limit as x approaches infinity is 0. Namely,

$$\lim_{x\to\infty}\left(\frac{5}{x^2}\right)=0$$

Therefore, we can rewrite our limit statement:

$$\lim_{x\to\infty}\left(\frac{3x^2}{x^2}\right)-\lim_{x\to\infty}\left(\frac{2x}{x^2}\right)+\lim_{x\to\infty}\left(\frac{5}{x^2}\right)=\lim_{x\to\infty}\left(\frac{3x^2}{x^2}\right)-0-0=\lim_{x\to\infty}\left(\frac{3x^2}{x^2}\right)$$

Now, we cancel out the x^2 terms to get $\lim\limits_{x\to\infty}(3)$.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{3x^2}{x^2}\right)&=\lim_{x\to\infty}(3)\\&=3\end{aligned}$$

Since the value 3 is independent of the value of x, the limit as x goes to infinity of 3 will always be 3. Therefore,

$$\lim_{x\to\infty}\left(\frac{3x^2-2x+5}{x^2}\right)=3 \tag{55}$$

A general trick to finding the end behavior of a rational function with the same degree is to divide the coefficient of the highest degree term of the numerator by that of the denominator. Namely, if n is the degree of both the numerator and the denominator (the greatest exponent),

$$\lim_{x\to\infty}\left(\frac{a^n+b^{n-\alpha}+\ldots}{c^n+d^{n-\beta}+\ldots}\right)=\frac{a}{c}$$

Where α and β are positive constants.

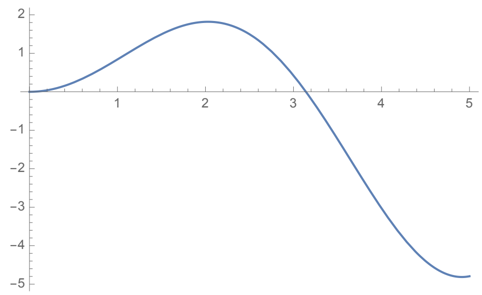
6.3 Intermediate Value Theorem

Now that we know about finding the end behavior of functions, there are things we can do inside a function over a finite domain. One of these things is called the Intermediate Value Theorem. The theorem only considers a function that is continuous over a certain domain $[x_1,x_2]$. If that initial condition is met, then the IVT states that if x-values a and b are in the domain and there is a y-value $f(n)$ that is between $f(a)$ and $f(b)$, then c is between a and b. Consider an example of a continuous function *f(x)*. If we know that $f(1)=4$ and $f(7)=6$, then there is some x-value, c, between 1 and 7 such that $f(c)=5$.

In addition, this theorem can be extended to consider the zeroes of a graph. If the function has a y-value $f(a)$ that is negative and a value $f(b)$ that is positive, then there is a certain x-value, c, between a and b such that $f(c)=0$. This makes sense; if the graph is continuous, it has to cross the x-axis at some point to go from negative to positive. For example, consider the function $f(x)=x\sin x$. First, we can make a table of various points on the graph:

x	1	2	3	4	5
y	0.841	1.819	0.423	-3.027	-4.795

As the table shows, the graph crosses the x-axis somewhere between $x = 3$ and $x = 4$. In addition, the graph $f(x) = x\sin x$ is continuous on the interval $[1,5]$. Because of this, we can confidently say that the graph $f(x) = x\sin x$ crosses the x-axis at least once on the interval $[3,4]$, because $f(x)$ goes from positive to negative. This also becomes apparent when we look at the graph of $f(x)$:

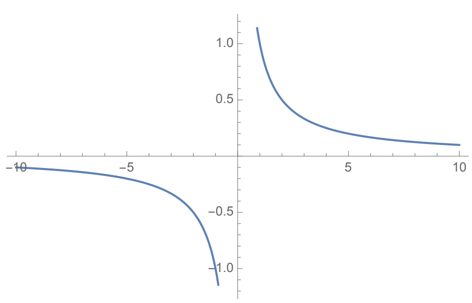


6.4 Transformations

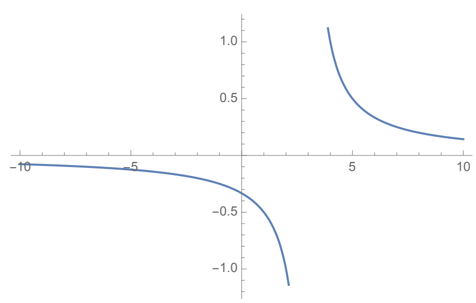
The purpose of this section is to provide a refresher for the transformations we can apply to graph. The three main transformations we will cover are translations, dilations, and reflections.

Translations

To start off, we will go over translations. Essentially, when you translate a graph, the same shape is kept, but the graph is just shifted with respect to the origin. We denote a horizontal translation as $f(x+\alpha)$, where α is the number of units the graph is shifted **to the left**. Logically, this means that if α is negative, the graph is shifted to the right. This notation may be confusing, so lets take a look at an actual function, $f(x) = \frac{1}{x}$. If we wanted to shift the graph 3 units to the right, the new function $g(x)$ would be $g(x) = f(x-3) = \frac{1}{x-3}$. Compare the two graphs:

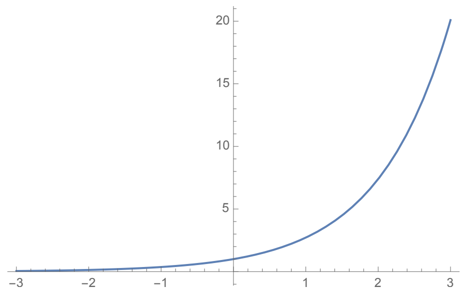


$f(x) = \frac{1}{x}$

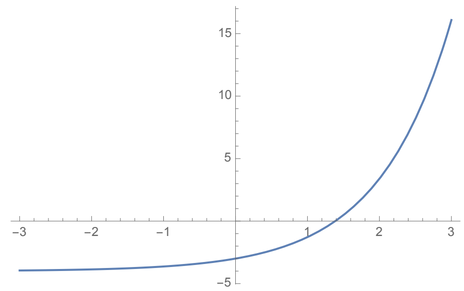


$g(x) = \frac{1}{x-3}$

If we wanted to vertically translate $f(x)$ by α units up, the new function would be $f(x)+\alpha$. Notice that α has nothing to do with x. Let's take the graph of $f(x) = e^x$ and shift it 4 units down:



$f(x) = e^x$



$g(x) = e^x - 4$

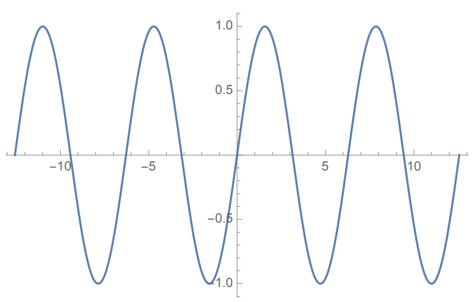
Again, the overall shape of the graph does not change, but the left end behavior does. Namely,

$$\lim_{x\to\infty^-} f(x) = 0 \qquad \lim_{x\to\infty^-} g(x) = -4$$

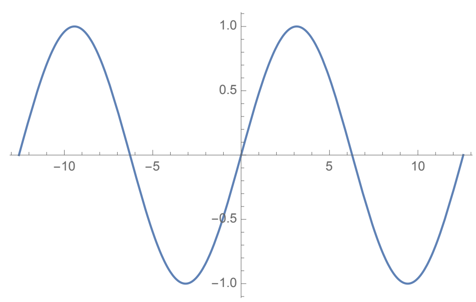
This is because as x goes to negative infinity, $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

Dilations

Horizontal dilations either stretch or compress a function. If we have the function $f(x)$ and want to horizontally stretch it by a factor of α, the new function would be $f(\frac{1}{\alpha}x)$. This means that the argument in f is decreased by a factor of α, assuming $\alpha > 1$. Because of this, larger x values must be taken to return the same value as or original $f(x)$. For example, consider the function $f(x) = \sin x$. If we wanted to horizontally by a factor of 2, the new function would be $f(\frac{1}{2}x)$, or $\sin(\frac{1}{2}x)$. Compare the two graphs:

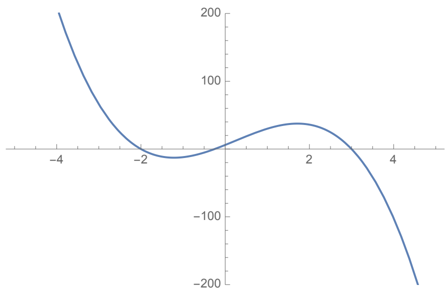


$f(x) = \sin x$

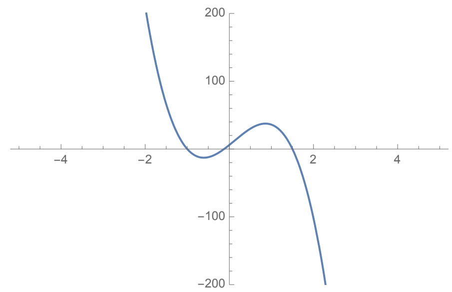


$f(\frac{1}{2}x) = \sin(\frac{x}{2})$

Logically, if we wanted to horizontally compress a function $f(x)$ by a factor of α, the new function would be $f(\alpha x)$, rather than $f(\frac{1}{\alpha}x)$. This is simply because a compression is the exact opposite of a stretch; stretching a function by a factor of α is the same as compressing it by a factor of $\frac{1}{\alpha}$. This is analogous to the fact that multiplication is the inverse operation of division. If we have a function $f(x) = -4x^3 + 3x^2 + 25x + 6$ and want to compress it by a factor of two, the new function would be $g(x) = f(2x) = -4(2x)^3 + 3(2x)^2 + 25(2x) + 6$:

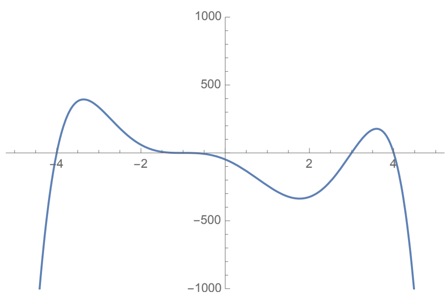


$f(x) = -4x^3 + 3x^2 + 25x + 6$

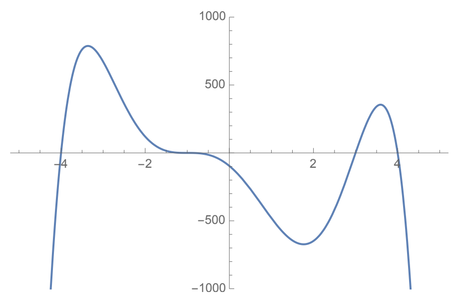


$g(x) = -4(2x)^3 + 3(2x)^2 + 25(2x) + 6$

To vertically dilate a function, we multiply the whole function by a factor, rather than just the argument (x). To vertically stretch a function $f(x)$ by a factor of α, we multiply the whole function by α. Namely, the new function is $\alpha f(x)$. This contrasts to horizontal dilations in two main ways. The first is that if we want to horizontally stretch a function by a factor of α, we multiply the argument by $\frac{1}{\alpha}$. However, for vertical stretches, our coefficient is α, rather than $\frac{1}{\alpha}$. In addition, for vertical dilations, the whole function is multiplied by a scalar, while in horizontal dilations, only the argument is multiplied by a scalar. Consider the function $f(x) = -x^6 + 22x^4 + 8x^3 - 93x^2 - 128x - 48$ if we want to vertically dilate it by a factor of 2, the new function would be $g(x) = 2f(x)$, or $2(-x^6 + 22x^4 + 8x^3 - 93x^2 - 128x - 48)$:



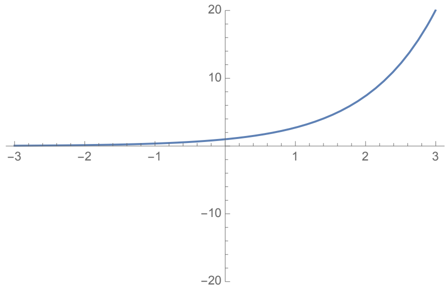
$f(x)$



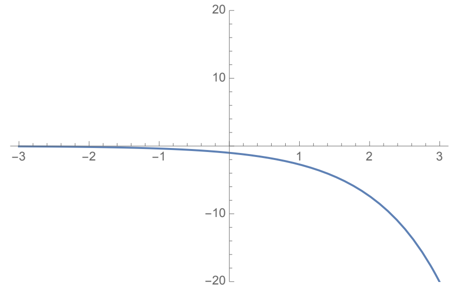
$g(x)$

Reflections

Although there are an infinite number of ways to reflect a graph, we will only focus on two ways: reflecting across the x-axis and the y-axis. To reflect an image across the x-axis, we essentially make every positive y value negative and vice versa, while keeping all x values the same. Mathematically, this means that $f(x)$ reflected across the x-axis is $-f(x)$. Consider the function $f(x) = e^x$. If we wanted to reflect it across the x-axis, our new function would be $g(x) = -f(x)$, or $-e^x$:

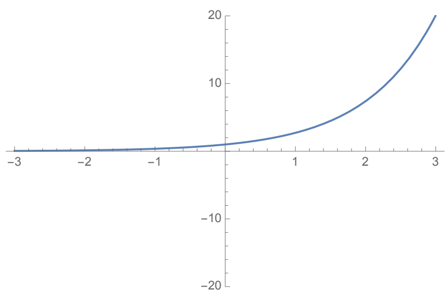


$f(x) = e^x$

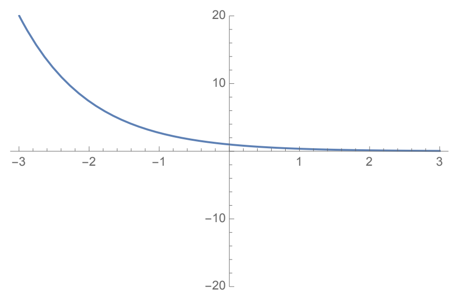


$g(x) = -e^x$

To reflect an image across the y-axis, we make every positive x value negative and vice versa, while keeping the y values the same. Mathematically, this means that the $f(x)$ reflected across the y-axis is $f(-x)$. Using the same example of $f(x) = e^x$, reflecting the function across the y-axis gives $g(x) = f(-x) = e^{-x}$:



$f(x) = e^x$



$g(x) = e^{-x}$

Summarizing all the transformations to $f(x)$, where $\alpha > 0$:

Horizontal Translation Right	$f(x-\alpha)$
Horizontal Translation Left	$f(x+\alpha)$
Vertical Translation Up	$f(x)+\alpha$
Vertical Translation Down	$f(x)-\alpha$
Horizontal Stretch	$f(\frac{1}{\alpha}x)$
Horizontal Compression	$f(\alpha x)$
Vertical Stretch	$\alpha f(x)$
Vertical Compression	$\frac{1}{\alpha}f(x)$
Reflection Across x-Axis	$-f(x)$
Reflection Across y-Axis	$f(-x)$

A Note on the Order of Transformations

The order in which transformations are made is essential to the resulting graph. For example, vertically dilating an image by a factor of α and then translating it β to the left is not the same as translating it and then dilating it. Thinking mathematically about this, if Θ represents the y-coordinate of a point, the dilated and then translated component, Θ', would be $\alpha\Theta+\beta$. However, if Θ is translated and then dilated, the new point would be $\alpha(\Theta+\beta)=\alpha\Theta+\alpha\beta$. Because of this, the order in which a problem tells you to transform a function does matter.

6.5 Function Operations and Compositions

Again, this section will serve mainly as a refresher of Algebra II concepts. If we have functions $f(x)$ and $g(x)$, there are several operations we can do involving **both** of these functions. This contrasts with most operations we have done in the past, because the haven't involved dealing with multiple functions.

Basic Function Operations

The additive property of functions states that $(f+g)(x)=f(x)+g(x)$. This is rather intuitive; adding two functions together gives a new function, which is the sum of the two original functions. Similarly, the subtractive property states $(f-g)(x)=f(x)-g(x)$

The multiplicative property follows a similar principle: $(f\bullet g)(x)=f(x)\bullet g(x)$. For example, if $f(x)=x+1$ and $g(x)=x^2$, $(f\bullet g)(x)=(x+1)(x^2)$. Functions can be divided as well: $(\frac{f}{g})(x)=\frac{f(x)}{g(x)}$

Compositions

Function compositions essentially make one function the argument of the other. Again as a refresher, the argument of a function is what is inside the parentheses: $f(\text{argument})$. In most cases x is the argument. However, with compositions, that is not the case; another function is the argument. We denote the composition of $f(x)$ and $g(x)$ as $f(g(x))$ or $(f\circ g)(x)$. These two statements both use $g(x)$ as the argument for $f(x)$. However, if we wanted $f(x)$ to be the argument for $g(x)$, we could rewrite our statement as $g(f(x))$ or $(g\circ f)(x)$. For example, consider the following functions: $f(x)=\frac{x}{x^2+1}$ and $g(x)=\sqrt{x}$.

$$f(g(x))=\frac{g(x)}{[g(x)]^2+1}=\frac{\sqrt{x}}{x+1}$$

$$g(f(x))=\sqrt{f(x)}=\sqrt{\frac{x}{x^2+1}}$$

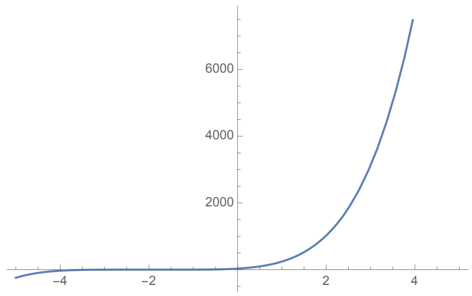
Functions are said to be inverses if $f(g(x))=g(f(x))=x$. How does this work, though? This is because if $f(x)$ is the inverse of $g(x)$ and vice versa, the composition of both functions, in either order, creates a new function (the identity function: $y=x$) that returns the argument. Let's consider the functions $f(x)=(x+2)^5$ and $g(x)=\sqrt[5]{x}-2$. If we wanted to verify that the two are inverses of each other, we simply do the composition test:

$$f(g(x))=(g(x)+2)^5=(\sqrt[5]{x}-2+2)^5=x$$

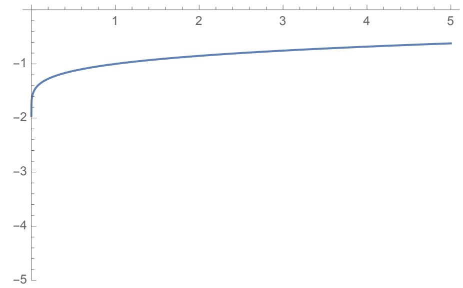
Ok, so the first part is all good. $f(g(x)) = x$. Now, we have to verify that $g(f(x)) = x$:

$$g(f(x)) = \sqrt[5]{f(x)} - 2 = \sqrt[5]{(x+2)^5} - 2 = x + 2 - 2 = x$$

By the composition test, we have determined that $f(x)$ and $g(x)$ are inverses of each other. $f(g(x)) = g(f(x)) = x$. Are $f(x)$ and $g(x)$ functions, though? We can graph them to see if they pass the vertical line test:



$f(x) = (x+2)^5$



$g(x) = \sqrt[5]{x} - 2$

Clearly, both graphs pass the vertical line test. This means that both $f(x)$ and $g(x)$ are functions, which makes them one-to-one functions. A one-to-one function is a function whose inverse is also a function.
Although this is most likely old knowledge, a function's inverse can be found by replacing the x values with y values and vice versa. For example, if we wanted to find the inverse of $f(x) = \frac{x+3}{x-8}$, we simply swap x and y:

$$\begin{aligned} x &= \frac{y+3}{y-8} \\ x(y-8) &= y+3 \\ xy - 8x &= y+3 \\ xy - y &= 8x+3 \\ y(x-1) &= 8x+3 \\ y = f^{-1}(x) &= \frac{8x+3}{x-1} \end{aligned}$$

6.6 Parametric Equations

A parametric equation essentially introduces a third variable (or parameter) to an equation. For most of this chapter, our new parameter will be t, where t means time. Usually, we have has the y-coordinate as a function of x, namely $f(x)$. Now, however, we will have two equations to represent an objects position - one for the y-coordinate and one for the x-coordinate. Both of these equations, $x(t)$ and $y(t)$, are functions of time. For example, consider a particle whose position can be modeled by the parametric equations

$$x(t) = t^2$$
$$y(t) = 3t - 6$$

This means that at time $t = 0$, the particle will have an x coordinate of 0 and a y coordinate of -6, which gives the particle coordinates $(0, -6)$ in the xy plane. Now, what happens when the particle starts moving? Another way to ask this is "What happens when t increases?" Well, lets make a table:

t	1	2	3	4	5
x	1	4	9	16	25
y	-3	0	3	6	9

The main advantage to parametric equations is that they allow us to tell where a particle is *at a given time*. If $t = 4$, the particle is at $(8, 6)$. An equation whose y-coordinate is only dependent on x is time-independent. We cannot answer the question, "Where is the particle after 7 seconds?" if t is not a parameter in the function.

What if we wanted to find y strictly as a function of x? Going back to the two parametric equations above, we can find what x is in terms of t:

$$x(t) = t^2$$

$$t = \sqrt{x(t)}$$

After we solve for t, we can substitute the x value back into the $y(t)$ equation to make it $y(x)$:

$$y(t) = 3t - 6$$

$$y(x) = 3\sqrt{x} - 6$$

All we did here was use the equation $t = \sqrt{x}$ and plug that into the $y(t)$ equation. The reason that $y(t)$ switched to $y(x)$ is because our new function is a function of x, rather than t.

6.7 Problems

1. Write a set of parametric equations for an object that follows a linear path, goes through the point $(3,8)$ at $t = 12$, and starts at the point $(-3,5)$.

2. The path of a rider on a ferris wheel can be modeled by the following equations

$$x(t) = 20\sin(\tfrac{t}{6})$$
$$y(t) = -20\cos(\tfrac{t}{6})$$

   (a) Where does the rider start on the ferris wheel? (Top, right, left, bottom)

   (b) Find $y(x)$

   (c) At what time is the rider at the highest point on the ferris wheel?

3. Graph the following parametric equations, drawing arrows on the graph to show how the particle moves as time increases:

   (a) $x(t) = t - 3$, $y(t) = 2t - 4$

   (b) $x(t) = 3 - 2t$, $y(t) = 1 + 2t$

   (c) $x(t) = 2\sin(t)$, $y(t) = 5\cos(t)$

   (d) $x(t) = 6\cos(t)\sin(t)$, $y(t) = 6\cos^2(t)$

4. Describe the following transformations, in order, that are applied to the functions:

   (a) $e^x \Rightarrow \frac{-5e^x}{e}$

   (b) $x! \Rightarrow \frac{x!}{x}$

   (c) $\frac{5x^2+2x}{x^3-1} \Rightarrow \frac{10(x+1)^2+4(x+1)}{2(x+1)^3-2} + 4$

5. Determine if a zero can be found in the following functions with the interval specified. Justify your answers using the IVT.

   (a) $f(x) = \sin x \cos x$ on $[1,2]$

   (b) $f(x) = \frac{e^{2x}}{x!} - 5$ on $[0,3]$

   (c) $f(x) = e^{\log_3 x} \sin x$ on $[0,4]$

6. Find the inverses of the following functions and determine if they are one-to-one. In addition, state the domain and range of the inverse function.

   (a) $f(x) = \frac{x+5}{x-5}$

(b) $f(x) = \sqrt[5]{x+2} - 3$

(c) $f(x) = 3e^{2x}$

(d) $f(x) = \ln|\tan^{-1}(\frac{x}{3})| - 2$

(e) $f(x) = \frac{x^2+x+1}{x}$

(f) ★ $f(x) = \sqrt[3]{x^3 + \frac{1}{x^3}}$ Note: before attempting this, you should be familiar with Cardano's method for solving a cubic equation.

(g) ★ $f(x) = x^{x^{x^{x^{x^{\cdots}}}}}$

(h) ★ $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$

7. What is $g(x)$ if $f(x) = 5x - 2$ and $g(f(x)) = 100x^2 - 80x$?

8. Prove that

   (a) the product of two even functions $f(x)$ and $g(x)$ is **even**

   (b) the product of two odd functions $f(x)$ and $g(x)$ is **even**

   (c) the product of an even function $f(x)$ and an odd function $g(x)$ is **odd**.

9. ★ AIME 1997 Problem 7: A car travels due east at $\frac{2}{3}$ miles per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{12}{\sqrt{2}}$ miles per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1 + t_2)$.

7 Exponential and Logarithmic Functions

This chapter will more deeply analyze the properties and applications of exponential and logarithmic functions. A logarithm is an expression that returns an argument; namely, the expression $a^b = c$ is equivalent to $\log_a c = b$. This is read as "log base a of c equals b." Another way of defining the logarithmic function is the inverse of the exponential function; consider the function $y = a^x$. If we wanted to find the inverse, we would first switch the x and y:

$$y = a^x \Rightarrow x = a^y$$

$$y = \log_a x$$

Therefore, the inverse of an exponential function is a log function and vice versa.

7.1 Exponential Functions

This section will provide a brief review of exponential functions. First off, however, we should go over the properties of exponents:

$$k^a k^b = k^{a+b} \tag{56}$$

$$\frac{k^a}{k^b} = k^{a-b} \tag{57}$$

$$(k^a)^b = k^{ab} \tag{58}$$

$$k^{-a} = k^{0-a} = \frac{k^0}{k^a} = \frac{1}{k^a} \tag{59}$$

Most other exponent properties can be derived from these two.

The general form for an exponential function is $f(x) = ab^x + c$, where a is a scalar constant and c is the vertical shift constant. If we wanted to horizontally shift a function k units to the right, our new function would be $f(x) = ab^{x-h} + c$. However, this simplifies to $a\frac{b^x}{b^h} + c = \frac{a}{b^h}b^x + c$. Thus, our original "general form" of the exponential function is retained; our new scalar constant is $\frac{a}{b^h}$. This is a beautiful property unique to exponential functions; multiplying the exponential term is equivalent to horizontally shifting the whole function.
With an exponential function in the form $f(x) = ab^x + c$, we can find the y-intercept in terms of a, b, and c. If $x = 0$, then $f(x) = ab^0 + c = a + c$. Therefore, the y-intercept is $a + c$. It is important to note, however, that this is only true when the function is in the form $f(x) = ab^x + c$. Practice at reducing an exponential function to the general form is especially useful for finding its properties. The x-intercept can be found by setting $f(x)$ to zero:

$$0 = ab^x + c$$

$$\frac{-c}{a} = b^x$$

$$x = \log_b\left(\frac{-c}{a}\right) = \log_b\left(\frac{a}{c}\right)$$

Note that the x-intercept is absolutely dependent on the value of c; if $c = 0$, the x-intercept is the log of 0, which does not exist. In addition, if c is positive, we end up taking the log of a negative (because $x = \log_b(\frac{-c}{a})$), so that does not exist either. Therefore, c must be negative in order for there to be an x-intercept on an exponential graph.

Exponential functions also have horizontal asymptotes, which we can define through end behavior. It is important to note that exponential functions in the form $f(x) = ab^x + c$ only approach the asymptote on one side;[5] Considering the end behavior of our original exponential function, $f(x) = ab^x + c$,

$$\text{Left end behavior: } \lim_{x\to\infty^-} f(x) = ab^{-\infty} + c = \frac{a}{b^\infty} + c = \frac{a}{\infty} + c = 0 + c = c$$
$$\text{Right end behavior: } \lim_{x\to\infty} f(x) = ab^{\infty} + c = \infty$$

Therefore, we have shown that the left end behavior of the function $f(x) = ab^x + c$ is c, but the right end behavior increases to infinity. If the function were $g(x) = ab^{-x} + c$, however, the end behavior would be the opposite. A function with an exponent of $-x$ is an exponential decay function, while a function with x is an exponential growth function:

$$\text{Left end behavior: } \lim_{x\to\infty^-} g(x) = ab^{-(-\infty)} + c = ab^{\infty} + c = \infty$$

$$\text{Right end behavior: } \lim_{x\to\infty} g(x) = ab^{-\infty} + c = \frac{a}{b^\infty} + c = \frac{a}{\infty} + c = 0 + c = c$$

This is simply due to the fact that b^x and b^{-x} are inversely related. As x gets bigger, b^x increases but b^{-x} approaches zero.
Now that we know the basic properties of exponential problems, let's take a look at an example:

Example 7.1

A deer population was measured every month for six months and the following numbers were recorded:

[5] Some exponential functions, however, approach a horizontal asymptote from both sides. For example, consider the function $f(x) = e^{1/x^2}$

Month	Population
0	35
1	40
2	50
3	70
4	110
5	190
6	350

Model the deer population as an exponential function, $P(t)$, where t is in months.

First off, we know the y-intercept is 35, or $a+c$, if the function follows the general form $f(x) = ab^x + c$. We also know that $P(1) = 40$, which is an increase in 5 deer from month 0. Because of this, we have the system of equations

$$35 = a + c$$
$$40 = ab^1 + c = ab + c$$

We can combine both of these equations:

$$35 + ab + c = 40 + a + c$$

$$ab = 5 + a$$

$$ab - a = 5$$

$$a(b-1) = 5$$

$$a = \frac{5}{b-1}$$

We also know that $P(2) = 50$, which means that $50 = ab^2 + c$. We can combine the equation with $35 = a + c$ to eliminate the c term:

$$35 = a + c$$
$$50 = ab^2 + c$$

$$35 + ab^2 + c = a + c + 50$$

$$ab^2 = a + 15$$

We can then combine this equation with $a = \frac{5}{b-1}$ by substituting every a value for $\frac{5}{b-1}$:

$$ab^2 = a + 15$$

$$\frac{5}{b-1}b^2 = \frac{5}{b-1} + 15$$

$$5b^2 = 5 + 15(b-1) = 5 + 15b - 15 = 15b - 10$$

Now, we can set up a quadratic:

$$5b^2 - 15b + 10 = 0$$

$$b^2 - 3b + 2 = 0$$

We now find that $b = 1$ or 2. However, b cannot be 1, given that $a = \frac{5}{b-1}$; the denominator would be zero. Now that we know $b = 2$, we can solve for a:

$$a = \frac{5}{b-1} = \frac{5}{2-1} = 5$$

Finally, we can solve for c.

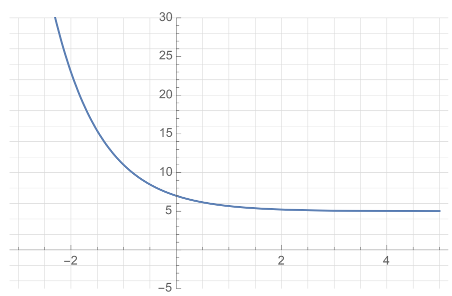
$$35 = a + c = 5 + c$$

$$c = 30$$

Now that we have the values for *a, b*, and c, we can plug them back into our equation to get:

$$\boxed{P(t) = 5(2^t) + 30}$$

If we are given a graph of an exponential equation, there are shortcuts we can take to find the equation more easily. If the end behavior of the graph is obvious, the c term is trivial. For example, say we want to find the equation of the following graph:



Although there is a concrete process we can do every time to find the exponential equation (as shown in the example above), no one wants to do that. First off, this graph is an exponential decay graph, which means it has the general form $f(x) = ab^{-x} + c$. In addition, the graph clearly shows a right end behavior of 5. Namely,

$$\lim_{x\to\infty} f(x) = ab^{-x} + c = 5$$

. As covered before, the right end behavior of an exponential decay graph is simply c. Thus, $c = 5$. Additionally, $f(0) = 7$. Knowing that $c = 5$, we obtain the equation $ab^0 = 2 \Rightarrow a = 2$. Knowing that $a = 2$ and $c = 5$, we can plug these into any other point on the graph to find b. Since $f(-1) = 11$, we can set up an equation and solve for b:

$$11 = ab^{-x} + c$$

$$11 = 2b^1 + 5$$
$$b = 3$$

Now, we plug in the values for *a, b,* and *c* to get

$$\boxed{f(x) = 2(3^{-x}) + 5}$$

7.2 Logarithms

Before we go over the properties and applications of logarithmic functions, it is important to have a solid understanding of the basic properties of logarithms; namely, the product, quotient, power, and change of base rule.

Power Rule

As stated before, the expression $\log_a b = c$ is equivalent to $a^c = b$. First, lets consider the following:

$$\log_a b = c \Rightarrow a^c = b$$

Multiplying both sides by constant k, we get

$$k \log_a b = ck$$

If we raise both sides of $a^c = b$ to k, we get

$$a^{ck} = b^k$$

Taking the log of the equation above,

$$\log_a b^k = ck$$

Since $k \log_a b$ also equals ck, we have proven the following identity:

$$\log_a b^k = k \log_a b \tag{60}$$

Product Rule

Before we delve into the product rule proof, it is essential to understand that $b^{\log_b a} = a$. If $f(x) = b^x$, then $f^{-1}(x) = \log_b x$. $b^{\log_b a}$ is just $f(f^{-1}(x))$, which, as we learned before, is x. beta Now, let $\log_a b = \beta$ and $\log_a c = \gamma$. Because of the property described above,

$$a^\beta = b \quad \text{and} \quad a^\gamma = c$$

We can then combine both equations to get

$$a^\beta a^\gamma = bc$$
$$a^{\beta+\gamma} = bc$$

Then, if we take the log of the equation above,

$$\log_a bc = \beta + \gamma$$

Substituting in for β and γ, we get the product rule of logarithms:

$$\log_a bc = \log_a b + \log_a c \tag{61}$$

Quotient Rule

While proving the quotient rule, we will take a similar approach to when we proved the product rule. Letting $\log_a b = \beta$ and $\log_a c = \gamma$ as before, we arrive at

$$a^\beta = b \quad \text{and} \quad a^\gamma = c$$

Now, instead of multiplying both equations together, we will divide:

$$\frac{a^\beta}{a^\gamma} = \frac{b}{c}$$

$$a^{\beta-\gamma} = \frac{b}{c}$$

Taking the log,

$$\log_a \frac{b}{c} = \beta - \gamma$$

Finally, substituting in for β and γ, we get the quotient rule of logarithms:

$$\log_a \frac{b}{c} = \log_a b - \log_a c \tag{62}$$

Change of Base Rule

If $\log_a b = c$, we know that $a^c = b$. We can take the logarithm of both sides of the equation to get

$$\log_k a^c = \log_k b \qquad \text{where } k \text{ is an arbitrary constant.}$$

This works based on the idea that if two things are equal, then their logs are equal. For example, if $a = b$, then $\log_k a = \log_k b$. Now, using the power rule, we get

$$c \log_k a = \log_k b$$

$$c = \frac{\log_k b}{\log_k a}$$

Substituting in for c,

$$\log_a b = \frac{\log_k b}{\log_k a}$$

It is important to understand that k does not have a specific value; it can be 5, 100, $\frac{1}{2}$, etc, as long as it is greater than zero and not 1. For example, we can express $\log_3 15$ as $\frac{\log_2 3}{\log_2 15}$, $\frac{\log_{300} 3}{\log_{300} 15}$, or $\frac{\log_{1/7} 3}{\log_{1/7} 15}$. They all equal the same thing.

In order to fully master these four log properties, you will need to solve plenty of problems. This isn't something where simply memorizing the formulas will work, as problems like these can take many different forms.

Example 7.2

Solve for x:

$$2\log_3 x - \log_3(x+6) = 1$$

Using logarithm properties,

$$\begin{aligned} 2\log_3 x - \log_3(x+6) &= \log_3 x^2 - \log_3(x+6) \\ &= \log_3 \frac{x^2}{x+6} \end{aligned}$$

Additionally, we know that $1 = \log_3 3$, because $3^1 = 3$. Our equality then becomes

$$\log_3 \frac{x^2}{x+6} = \log_3 3$$

Taking out the log from both sides,

$$\frac{x^2}{x+6} = 3$$

Then, we just use algebra to solve the quadratic, getting

$$= -3, 6$$

However, there is a catch; if we plug $x = -3$ back into the original function, we find that $2\log_3(-3)$ is undefined! Because of this, we only have one solution:

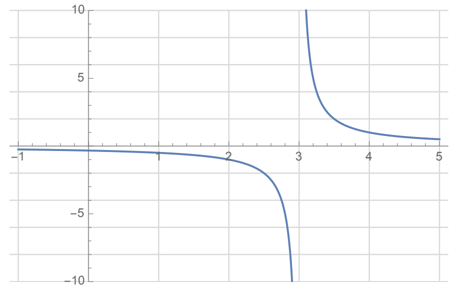
$$\boxed{x = 6}$$

7.3 Rational Functions

Rational functions are defined as functions that are the ratios of polynomials. This means a fraction of two polynomial functions. If $f(x)$ and $g(x)$ are polynomial functions, then $r(x) = \frac{f(x)}{g(x)}$ is a rational function. For example, a rational function could be

$$f(x) = \frac{3x^2 + 2x - 5}{x^4 + 2x^3 - 7x^2 + 8}$$

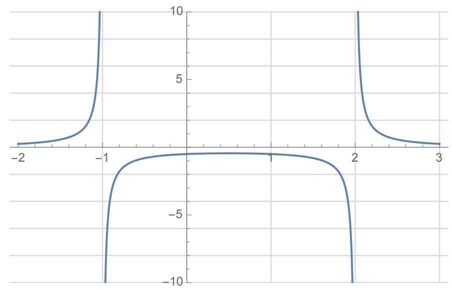
The most general form of a rational function is $f(x) = \frac{1}{x}$. Vertical asymptotes are one of the features unique to rational functions (with the exception of logarithmic functions). The x coordinate of the vertical asymptote is the x value that makes the denominator zero. This makes sense; as the value of the denominator approaches zero, the absolute value of the function increases. However, the function is not defined when the denominator is zero, which results in it growing infinitely without bound. Let's first start off by taking a look at the function $f(x) = \frac{1}{x-3}$.



Obviously, looking at the graph wasn't necessary to see that we have a vertical asymptote at $x = 3$. However, the graph also shows the same result. As x approaches 3, $f(x)$ gets larger and larger. Rational functions can also have more than one vertical asymptote. Take the function

$$f(x) = \frac{1}{(x+1)(x-2)}$$

. The denominator is zero when $x = -1$ or 2. The graph shows the asymptotes:



Similar to every other function, we can apply transformations to rational functions in the same manner. For example, if we want to shift $f(x) = \frac{1}{x}$ four units to the left, our new function is $g(x) = f(x+4)$, or $\frac{1}{x+4}$. Dilations and vertical shifts follow the same rule. The table below illustrates all the basic transformations that can be applied to the parent rational function, $\frac{1}{x}$, where $\alpha > 0$.

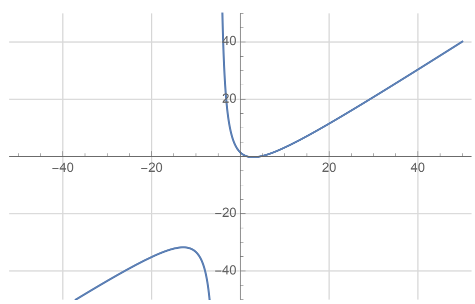
Horizontal Translation Right	$f(x-\alpha) = \frac{1}{x-\alpha}$
Horizontal Translation Left	$f(x+\alpha) = \frac{1}{x+\alpha}$
Vertical Translation Up	$f(x)+\alpha = \frac{1}{x}+\alpha$
Vertical Translation Down	$f(x)-\alpha = \frac{1}{x}-\alpha$
Horizontal Stretch	$f(\frac{1}{\alpha}x) = \frac{\alpha}{x}$
Horizontal Compression	$f(\alpha x) = \frac{1}{\alpha x}$
Vertical Stretch	$\alpha f(x) = \frac{\alpha}{x}$
Vertical Compression	$\frac{1}{\alpha}f(x) = \frac{1}{\alpha x}$
Reflection Across x-Axis	$-f(x) = -\frac{1}{x}$
Reflection Across y-Axis	$f(-x) = \frac{1}{-x}$

This table shows a beautiful symmetry to the function $f(x) = \frac{1}{x}$; reflecting $\frac{1}{x}$ across the x-axis produces the exact same result as if it were reflected across the y-axis. In addition, a vertical stretch by a factor of α to $\frac{1}{x}$ is the exact same as a horizontal stretch. The same property applies to vertical and horizontal compressions.

Most of the time a rational function will not be as easy to understand as the simple $\frac{1}{x}$. For example, consider the function $f(x) = \frac{x^2-6x+7}{x+5}$. Before we do anything, we must make sure the degree of the numerator is less than the denominator. Once this is done, we can better understand the asymptotes and end behavior of the function. To get the degree of the numerator less than the denominator, we must do polynomial division:

$$\begin{array}{r|l} & \quad\quad\quad\; x-11 \\ x+5 & \; x^2 \;\; -6x \;\; +7 \\ & -x^2 \;\; -5x \\ \hline & \quad\; -11x \;\; +7 \\ & \quad\quad 11x+55 \\ \hline & \quad\quad\quad\quad 62 \end{array}$$

Our remainder is 62. We can rewrite $f(x)$ as $x-11+\frac{62}{x+5}$. Now, notice that as x gets very large, the fraction $\frac{62}{x+5}$ approaches zero. This is to say that as $|x|$ gets larger, $f(x)$ approaches the behavior of just $x-11$. We call $x-11$ the oblique asymptote, or slant asymptote, of $f(x)$. When $f(x)$ is graphed, our oblique asymptote is shown:



Clearly, the function is approaching a certain behavior as $|x|$ increases.

Some rational functions have the same term in a numerator and denominator. For example, consider the function $f(x) = \frac{x^2+x-6}{x^2-4}$. Factoring, we get $f(x) = \frac{(x+3)(x-2)}{(x+2)(x-2)}$. Cancelling out terms, we get $f(x) = \frac{x+3}{x+2}$. Now, this function isn't too hard to graph; we do polynomial division, find the asymptotes, and graph. However, there is a catch. $f(x)$ is not defined at $x = 2$. Although we manipulated $f(x)$ to be defined at 2, the original function, $\frac{x^2+x-6}{x^2-4}$, has a denominator of zero when $x = 2$. Because of this, we have a "hole" at $x = 2$. If we wanted to find the y-coordinate of the hole, we would plug $x = 2$ into $f(x) = \frac{x+3}{x+2}$ to find the ordered pair: $(2, \frac{5}{4})$. Remember, the graph of $f(x) = \frac{x+3}{x+2}$ is the exact same as the graph of $f(x) = \frac{x^2+x-6}{x^2-4}$, with the exception of the hole. It is important to note that most of the time, holes aren't as easy to tell as the example above. When evaluating any rational function, it is always a good idea to factor it and see if you have any holes.

What if we have a complex rational function like $f(x) = \frac{2x^3+12x^2+6x-20}{3x^2+6x-9}$? First, let's use polynomial division to get the degree of the numerator less than the denominator:

$$
\begin{array}{r|l}
 & \frac{2}{3}x \quad + \frac{8}{3} \\
\hline
3x^2+6x-9 & 2x^3+12x^2 \quad +6x-20 \\
 & -2x^3 \quad -4x^2 \quad +6x \\
\hline
 & \qquad 8x^2+12x-20 \\
 & \qquad -8x^2-16x+24 \\
\hline
 & \qquad\qquad -4x \quad +4
\end{array}
$$

We can then rewrite the function as $f(x) = \frac{2}{3}x + \frac{8}{3} + \frac{-4x+4}{3x^2+6x-9}$.

Before we do anything else with $f(x)$, we can determine its oblique asymptote: $\frac{2}{3}x + \frac{8}{3}$. Again, this means that as $|x|$ increases, the function gets more and more linear, approaching the shape of $\frac{2}{3}x + \frac{8}{3}$.

Factoring the denominator, we get

$$f(x) = \frac{2}{3}x + \frac{8}{3} + \frac{-4(x-1)}{3(x+3)(x-1)}$$

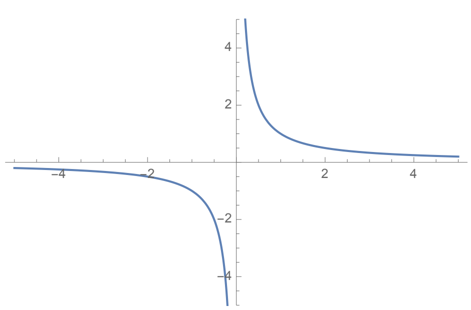
Now, it becomes obvious that we have a hole at $x = 1$. If we want to find the y-coordinate of the hole, we can cancel out the $(x-1)$ terms and plug in $x = 1$:

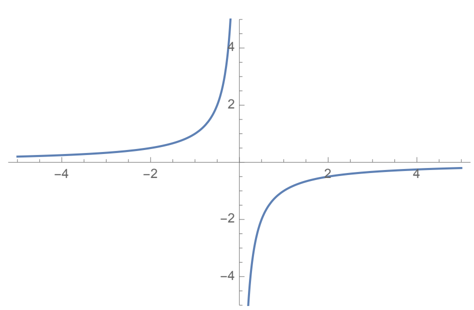
$$f(x) = \frac{2}{3}x + \frac{8}{3} + \frac{-4}{3(x+3)} \Rightarrow f(1) = \frac{2}{3} + \frac{8}{3} + \frac{-4}{12} = 3$$

Therefore, the coordinate of our hole is $(1,3)$. Going back to our function,

$$f(x) = \frac{2}{3}x + \frac{8}{3} + \frac{-4}{3(x+3)}$$

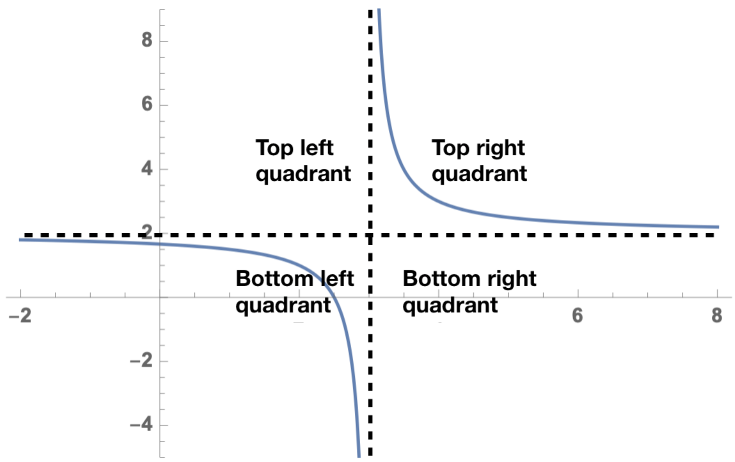
we can see that that there is a vertical asymptote at $x = -3$, because the denominator is zero when $x = -3$. Now that we know the asymptotes of the function, we have a general idea of what it looks like. Since there is only one vertical asymptote and the coefficients of the highest degree terms in the fraction are negative and positive (negative in the numerator in positive in the denominator), we know where the graph lies relative to the asymptotes. For example, if the coefficient of the highest degree terms in the fraction are positive and negative (such as $-\frac{1}{x}$), then the graph will exhibit a distinct behavior *relative to the asymptotes* when compared to $\frac{1}{x}$:





$f(x) = \frac{1}{x}$ $f(x) = -\frac{1}{x}$

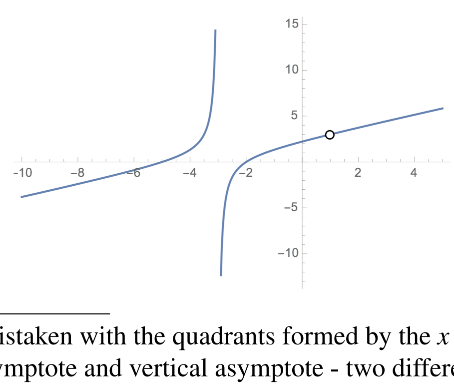
The graph is essentially "flipped" with respect to the asymptotes. Since one of the coefficients is positive while the other is negative, we know that the graph will behave in the similar way (with respect to the asymptotes) as $f(x) = -\frac{1}{x}$; the graph will lie in the top left and bottom right "quadrants"[6] formed by the asymptotes. See the figure below for an example of the quadrants I am referring to (the function is $f(x) = \frac{1}{x-3} + 2$):



After this, we have a pretty good idea of what the graph would look like. There isn't really a whole lot else we can do, actually, besides plugging in points to make sure the properties of the graph we have found are correct. If we plug in $x = 4$ to $f(x)$, then we know it has to be below the line $f(x) = \frac{2}{3}x + \frac{8}{3}$. This is because it is to the right of the vertical asymptote, so it has to be below the oblique asymptote to be in the "bottom right" quadrant described above. Plugging in $f(4)$, we get

$$f(4) = \frac{2}{3}(4) + \frac{8}{3} + \frac{-4}{3(x+3)}$$

Straight off the bat, we can tell that $f(4)$ is below the oblique asymptote. $\frac{-4}{3(x+3)}$ will always be negative as long as $(x+3)$ is positive; in this case it is, because $x = 4$. So, our property is correct. We can also plug in a point x that is less than 3 to make sure it is above the oblique asymptote. Then, we can draw our asymptotes and then plot $f(x)$:

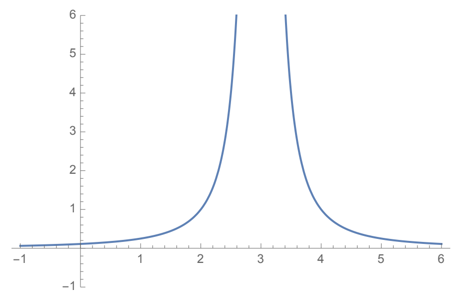


[6]These quadrants are not to be mistaken with the quadrants formed by the x and y axes; they are just the quadrants formed by the horizontal/oblique asymptote and vertical asymptote - two different lines

This is definitely one of the more complicated rational functions to graph. Besides this, there are only a few more types of rational functions. One of them is if the denominator is squared. For example, consider the function

$$f(x) = \frac{1}{(x-3)^2}$$

. Obviously, we will have a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 3$. But what around the asymptotes? The main concept to understand is that the function is always positive. This is because any real number squared is always positive. The graph is similar to $f(x) = \frac{1}{x-3}$, except the part in the bottom-left "asymptote quadrant" has been flipped up to the top right, because it is now positive. In addition, the curve is more steep than $f(x) = \frac{1}{x-3}$, because the denominator is squared. This results in the denominator approaching zero or infinity at a quicker rate than just $x-3$; another way of saying this is that $(x-3)^2$ "grows" faster than $x-3$.



The last type of rational functions we will cover are those with more than one vertical asymptote. For example, consider the function

$$f(x) = \frac{x-2}{(x+2)(x-3)}$$

Obviously, we will have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = -2$ and $x = 3$. How does it behave inside those asymptotes, though? Partial fractions[7] help us understand the behavior; we can decompose $f(x)$ into two fractions:

$$\frac{x-2}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$x-2 = A(x-3) + B(x+2)$$

$$3-2 = A(3-3) + B(3+2)$$

$$B = \frac{1}{5}$$

$$-2-2 = A(-2-3) + B(-2+2)$$

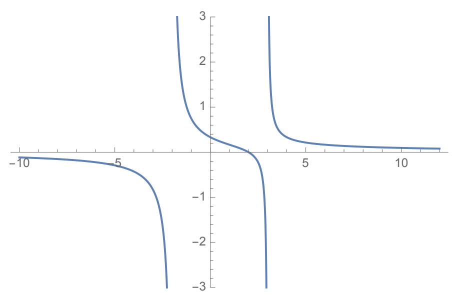
$$A = \frac{4}{5}$$

[7]Partial fractions are discussed in the "Miscellaneous Topics" chapter. See page 148

We can now rewrite $f(x)$:

$$\frac{4}{5(x+2)} + \frac{1}{5(x-3)}$$

This helps us better understand what happens around each asymptote of the function. For example, if we choose a value of x close to the asymptote, say 3.1, then our $\frac{1}{5(x-3)}$ term will have much more influence on the function because the denominator is much closer to zero then $\frac{4}{5(x+2)}$. In fact, as x approaches 3 from the positive side, $\frac{1}{5(x-3)}$ gets larger and larger approaching infinity. This means that relative to the vertical asymptote at $x = 3$ and the horizontal asymptote at $y = 0$, the graph is in the top right and bottom left quadrant. Now, let's look at how it behaves around the vertical asymptote at $x = -2$. Plugging in $x = -1.9$, we see that $f(x) = \frac{4}{5(.1)} + \frac{1}{5(-4.9)}$. Again, because $\frac{4}{5(.1)}$ has way more say in what the graph does than $\frac{1}{5(-4.9)}$, we only have to see if $\frac{4}{5(.1)}$ is positive or negative. Obviously it is positive, so we see that the graph is in the top right and bottom left quadrant relative to the vertical asymptote at $x = -2$ and the horizontal at $y = 0$. How does the graph go from a bottom quadrant to a top, though (bottom quadrant around vertical asymptote at $x = -2$ and top quadrant at $x = 3$)? Let's look at the graph:



It simply goes through the x-axis. If we look at the behavior around an asymptote, it really does look like a single-asymptote rational function. This is just because the other fraction plays such a small role in the value of the function because the denominator is far from zero.

7.4 Problems

1. Solve for x

   (a) $4^x = 8^{2x-1}$

   (b) $3^{(16x^2-20x+4)} = 81^{(x-1)}$

   (c) $\frac{2+\log(\log(x^{10}))}{2\log(\log(x^{1000}))} = x$

   (d) $\log_8 x + \log_8 \frac{1}{6} = \frac{2}{3}$

   (e) $\log\left(\log 3 + \frac{\log(x+1)}{\log 3}\right) = 0$

   (f) $\ln(x^2+x) = \ln x^2 + \ln x$

   (g) $\ln[x^{\ln x}] = e$

   (h) ★ $\ln\left(x^{x^{x^{x^{\cdots}}}}\right) = e$ Hint: This is a defective problem. Explain why there is no solution, even though you may have gotten a numerical value.

   (i) ★ $x^{(x^3)} = 3$ Hint: this can be solved **very** easily with clever substitution

   (j) ★ $(2+\log x)^3 + (-1+\log x)^3 = (1+\log x^2)^3$

2. Graph the following rational functions

   (a) $y = \dfrac{1}{(x-1)^2} + \dfrac{1}{(x+3)}$

   (b) $y = \dfrac{2}{x} - \dfrac{1}{(x-2)}$

   (c) $y = \dfrac{3x}{x^2+2x-3}$

   (d) $y = \dfrac{x^3+2x^2-x-2}{x^2+6x+8}$

   (e) $4x^2 - x^2y - 2xy + 5 = 0$

3. Given the logarithmic function $f(x) = \ln x^3 + 2$, give functions that would provide the following transformations:

   (a) Shift 3 units up and 2 units to the right

   (b) Dilate by a factor of 3 and then shift 5 units to the right

   (c) Reflect over the y-axis and then the x-axis

   (d) Reflect over the line $x = y$

4. Data regarding a bacteria population was collected, shown in the table below. The population grows exponentially.

Time (hours)	Population (millions)
1	9
2	21
3	57
4	165
5	489
6	1,461

(a) Find an equation that models the population as a function of time t

(b) What was the initial population?

(c) Suppose at time $t = 3$, the scientists introduced a chemical into the culture that killed the bacteria, and the population decayed exponentially:

Time (hours)	Population (millions)
1	9
2	21
3	57
4	21
5	9

Write a piecewise function that represents the population at time t.

5. If $\log_{12} 27 = a$, then find $\log_6 16$ in terms of a.

6. When Daenerys is mining Dragonglass, she realizes that it radioactively decays! She mined 250g of it, and 30 minutes later, only 175g remained. Assuming the amount of Dragonglass $D(t)$ decays exponentially, find $D(t)$ in the form $D(t) = Ae^{kt}$.

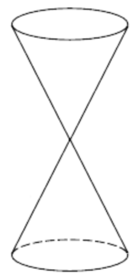
7. ★Solve the following system of logarithmic equations

$$xy = c^2$$

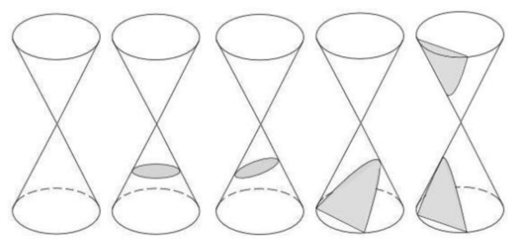
$$\left(\frac{\log x}{\log 2}\right)^2 + \left(\frac{\log y}{\log 2}\right)^2 = \frac{5}{2}\left(\frac{\log c}{\log 2}\right)^2$$

8 Conic Sections

This chapter will cover all the relations (not functions) that we can derive from a cone. Not any ordinary cone, though - a double cone:



We can "slice" this cone in different ways to produce different shapes. As it turns out, these different shapes actually have properties and can be graphed without any rigorous computations.



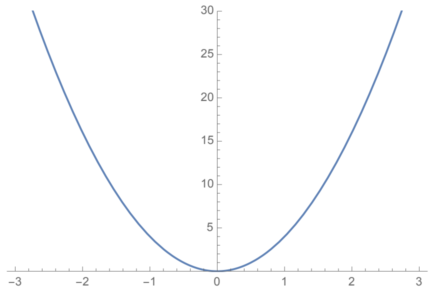
The focus of this chapter will be the functions that are produced from slicing a cone in these four different ways. We will explore the properties of the functions of circles, ellipses, parabolas, and hyperbolas, with an introduction to 3-dimensional functions. Out of all the units in Pre-Calculus, conics will probably take the most time to fully master.

8.1 Parabolas

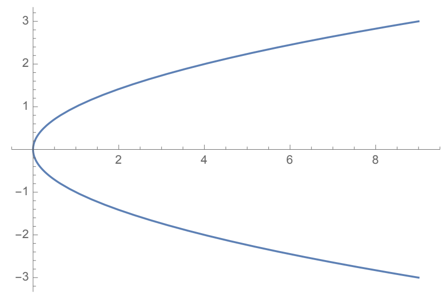
We will start off with the shape that is probably most familiar - the parabola. In this chapter, we will define a parabola as all the points in a plane that are equidistant from both the directrix and the focus. Now, what are the directrix and focus? Consider a parabola in standard form: $(y-k) = a(x-h)^2$. This can also be written as $4p(y-k) = (x-h)^2$, where $a = \frac{1}{4p}$. We define the focus as a point with coordinates $(h, k+p)$ *if the parabola opens up vertically*. If the parabola were to open horizontally, like $4p(x-h) = (y-k)^2$, then the focus would have coordinates $(h+p, k)$. p is just the distance from the focus to the vertex. The directrix of a parabola is a line defined by the equation $y = k-p$ if the graph opens vertically. If it opens horizontally, the equation is $x = h-p$. The table below synthesizes the equations for vertically and horizontally opening graphs.

	Opens Vertically	Opens Horizontally
Equation	$4p(y-k) = (x-h)^2$	$4p(x-h) = (y-k)^2$
Focus	$(h, k+p)$	$(h+p, k)$
Directrix	$y = k-p$	$x = h-p$

Just for clarification, refer to the following graphs of a vertically opening parabola and a horizontally opening parabola:



Vertically opening



Horizontally opening

Before we go on, let's do a practice problem.

Example 8.1

Given the equation $2x = y^2 - 4y + 2$ find

(a) the direction the parabola opens in

(b) the focus

(c) the directrix

First, we want to put the equation in standard form. Because of this, it will be in the form $4p(x-h) = (y-k)^2$. Because the parabola has a y^2 term, it will **open horizontally**.
We can first complete the square on the right side to get

$$2x + 2 = y^2 - 4y + 4$$

Now, we can put the equation in standard form; notice that $(y-2)^2 = y^2 - 4y + 4$:

$$2(x+1) = (y-2)^2$$

Now that we have the correct form for our parabola, we can find p:

$$2 = 4p \Rightarrow p = \frac{1}{2}$$

Before we do anything else, it would be a good idea to make a quick note of all the important values in the equation, so we can plug them in easily to find the focus and directrix:

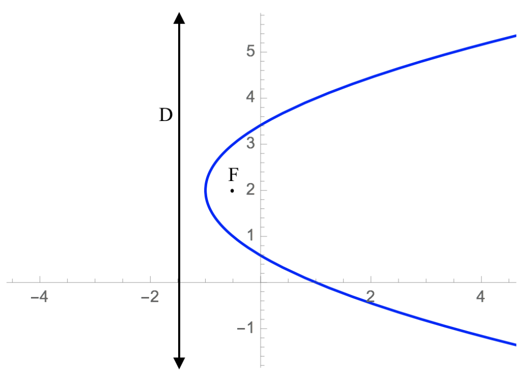
$$h = -1$$
$$k = 2$$
$$p = \frac{1}{2}$$

Since the parabola opens horizontally, we use the equations $(h+p,k)$ and $x = h-p$ to find the focus and directrix, respectively.

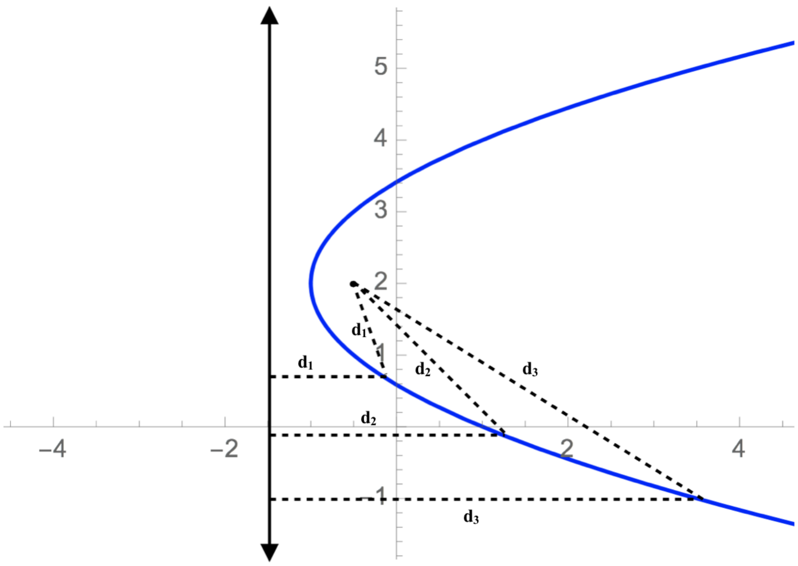
$$\text{Focus:} \quad (-1+\frac{1}{2},2) = (-\frac{1}{2},2)$$

$$\text{Directrix:} \quad x = -1-\frac{1}{2} \Rightarrow x = -\frac{3}{2}$$

The graph, with directrix D and focus F, is pictured below.

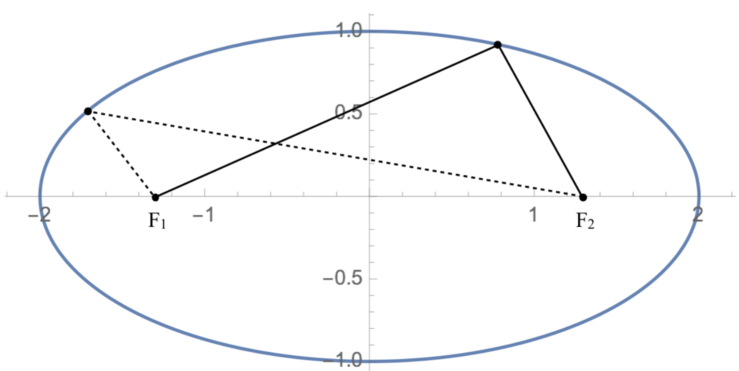


The main concept behind the focus and directrix is that they *define* the parabola. The infinite collection of points is the set that is equidistant from both the focus and directrix. Now, how do we define distance from a line? We take the point on the line that makes the connecting line perpendicular to the directrix. This is illustrated with the graph we just did, where d_1, d_2, and d_3 are distances:



8.2 Ellipses

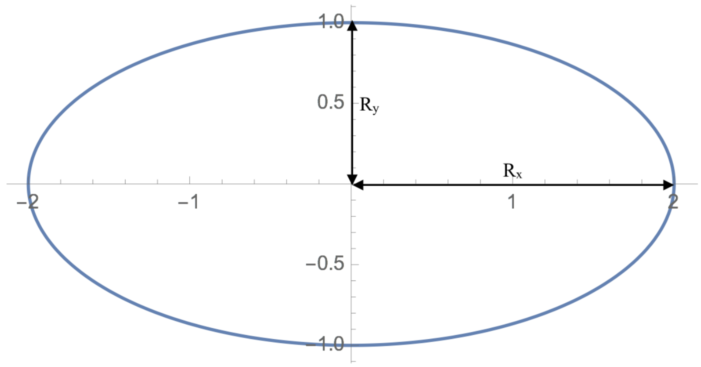
An ellipse is defined as a regular oval shape, traced by a point moving in a plane so that the sum of its distances from two other points (the foci, plural for focus) is constant. This definition is illustrated below, where the sum of the lengths of the dashed lines is equal to the sum of the lengths of the solid lines:



As it turns out, we can describe an ellipse with a set of parametric polar equations:

$$x(\theta) = R_x \cos\theta$$
$$y(\theta) = R_y \sin\theta$$

R_x and R_y represent the maximum x and y that the graph can be away from the origin; the amplitude of each component:



The ellipse is said to have vertices at $(\pm R_x, 0)$ and co-vertices at $(0, \pm R_y)$ if $R_x > R_y$. If $R_y > R_x$, then the ellipse has vertices at $(0, \pm R_y)$ and co-vertices at $(\pm R_x, 0)$. In other words, vertices are always lawger than co-verticies.

Now, what if we wanted to find an equation that only involved x and y, we would have to solve for θ in the x equation and then plug that into the equation for y. We could also do the opposite, solving for θ in the y equation and then plug it into the x equation.

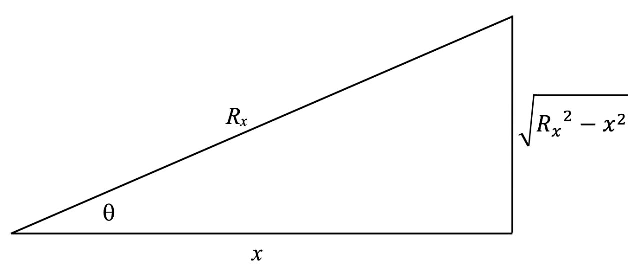
$$\begin{matrix} x(\theta) = R_x \cos\theta \\ y(\theta) = R_y \sin\theta \end{matrix} \Rightarrow \frac{x}{R_x} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{x}{R_x}\right)$$

Plugging in θ to the $y(\theta)$ equation,

$$y = R_y \sin\left[\cos^{-1}\left(\frac{x}{R_x}\right)\right]$$

Now, since we know that $\cos(\frac{\text{adjacent}}{\text{hypotenuse}}) = \theta$, we can set up a right triangle and solve for $\sin\theta$, which is $\frac{\text{opposite}}{\text{hypotenuse}}$



Using Pythagorean theorem, we know that the opposite side is $\sqrt{{R_x}^2 - x^2}$. We can now say that $\sin\left[\cos^{-1}\left(\frac{x}{R_x}\right)\right] = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{{R_x}^2 - x^2}}{R_x}$. Therefore,

$$y = R_y \frac{\sqrt{{R_x}^2 - x^2}}{R_x}$$

Squaring both sides,

$$y^2 = {R_y}^2 \left(\frac{{R_x}^2 - x^2}{{R_x}^2} \right)$$

$$\frac{y^2}{{R_y}^2} = \frac{{R_x}^2}{{R_x}^2} - \frac{x^2}{{R_x}^2}$$

$$\frac{y^2}{{R_y}^2} + \frac{x^2}{{R_x}^2} = 1$$

We will substitute $a = R_x$ and $b = R_y$ if $R_x > R_y$. If $R_y > R_x$, then we will let $a = R_y$ and $b = R_x$. In other words, a will always be greater than b. We do this simply because it allows us to compare the ellipse to other conic sections easier:

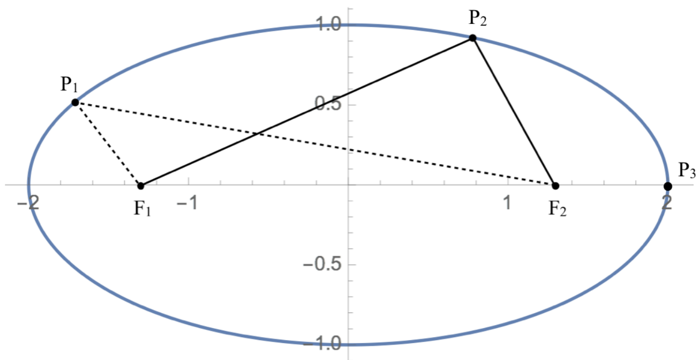
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{for } R_x > R_y$$

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \qquad \text{for } R_y > R_x$$

These are our general equations for an ellipse centered around the origin. Obviously, the basic vertical and horizontal transformation properties apply. Namely, the equation for a horizontal ellipse centered around the point (h,k) is:

$$\frac{(x-k)^2}{a^2} + \frac{(y-h)^2}{b^2} = 1 \tag{63}$$

Now, how do we find the foci of the ellipse? Obviously, this depends on its orientation. If $R_x > R_y$, for example, then the foci will be on the horizontal axis of symmetry of the ellipse. However, if $R_x < R_y$, then the foci will be on the vertical axis of symmetry. Before we define the foci of the ellipse, however, we will first define a value c. If $R_x > R_y$, then $c = \sqrt{{R_x}^2 - {R_y}^2}$. Similarly, if $R_x < R_y$, then $c = \sqrt{{R_y}^2 - {R_x}^2}$. This shouldn't be too hard to remember, because if we evaluated c in the wrong order, we would end up with the square root of a negative. We can rewrite c as $\sqrt{a^2 - b^2}$, since a and b will differ based on the major axis. If $R_x > R_y$, then the foci of an ellipse with center (h,k) are located at $(h \pm c, k)$. For $R_x < R_y$, the foci are at $(h, k \pm c)$. This makes sense. Given arbitrary points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the horizontal ellipse shown below, we can prove the location of the foci. The main idea is the total length of the dotted lines is equal to the total length of the solid lines. However, if we were to consider a point P_3 centered on the vertex, the total distance to F_1 and F_2 would be $R_x + c$ and $R_x - c$, respectively. The sum of these distances is $R_x + c + R_x - c = 2R_x$. Because the sum of the distances from foci is constant for all points on an ellipse, we now know that on an ellipse with a horizontal major axis, the total distance from the foci will always be $2R_x$. This can be generalized to any ellipse, regardless of the orientation of the major axes. For any point on an ellipse, the sum of the distances to the foci will always be $2a$.



If point P_1 has the coordinates (x,y), we can express the total length of the dotted lines, ℓ_{dotted}, as a sum of the two distances between the point and the two foci:

$$\ell_{dotted} = \sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2}$$

We also know that $\ell_{dotted} = 2a$, because we proved that above.

$$\sqrt{(x+c))^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

Rearranging terms,

$$\sqrt{(x+c))^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

Squaring both sides,

$$(x+c)^2+y^2 = [2a - \sqrt{(x-c)^2+y^2}]^2$$

$$x^2+2cx+c^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

Cancelling and rearranging,

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2+y^2}$$

Squaring both sides,

$$c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

Cancelling and rewriting,

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$x^2(a^2-c^2) + a^2y^2 = a^2(a^2-c^2)$$

Now, we will substitute in for a^2-c^2 by letting $b^2 = a^2-c^2$

$$x^2b^2 + a^2y^2 = a^2b^2$$

Finally, dividing by a^2b^2,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We have formally derived our final equation for an ellipse with a horizontal major axis. The derivation would be similar for an ellipse with a vertical major axis, except our equation would be $\frac{y^2}{a^2}+\frac{a^2}{b^2}=1$ instead. Additionally, because we let $b^2=a^2-c^2$, we know that $c^2=a^2-b^2$. Because of this, the foci are located a distance c, or $\sqrt{a^2-b^2}$ away from the center of the ellipse.

Example 8.2

Graph the following ellipse:

$$\frac{(y-1)^2}{2}+\frac{(x+3)^2}{8}=2$$

First, we must get the equation into our general ellipse form. Dividing by 2,

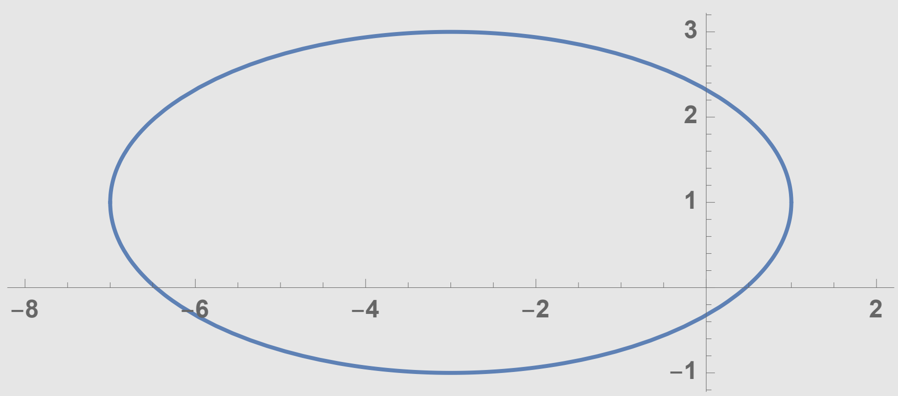
$$\frac{(y-1)^2}{4}+\frac{(x+3)^2}{16}=1$$

Now, since the denominator for the x term is greater then that of y, we know that the major axis will be parallel to the x-axis, and also that $a^2=16$. Additionally, we know that $b^2=4$, which gives us

$$a=4$$

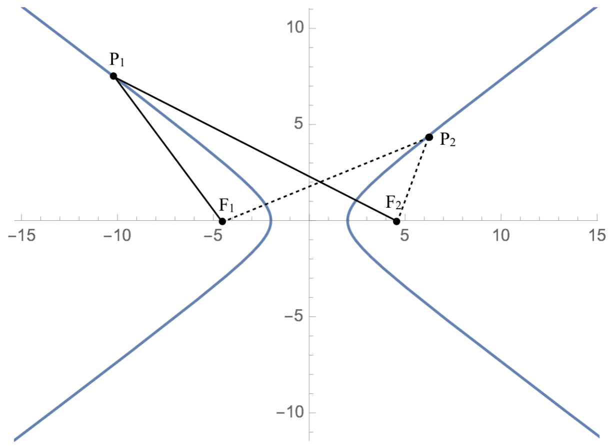
$$b=2$$

Again, this means that the distance from the center to a vertex is 4, and that the distance from the center to a co-vertex is 2. Additionally, because of the $(y-1)$ and $(x+3)$ terms, we can see that the center of our ellipse is at $(-3,1)$. Now that we have all this information, the only thing left is to graph:



8.3 Hyperbolas

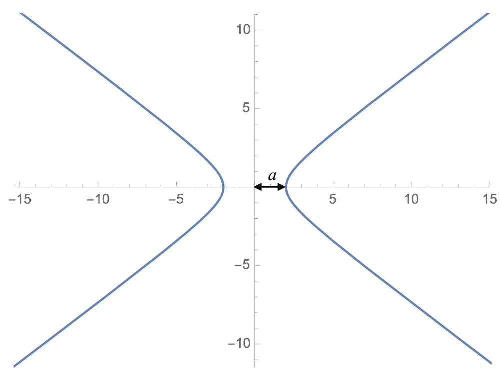
Similar to an ellipse, a hyperbola is defined as all points in a plane whose distances to two foci in the plane have a constant difference. This is the same definition as an ellipse except that it is the difference, rather than a sum.



Based on the image above, the definition of the hyperbola states that the difference in the distances of the solid lines is equal to the difference in the distances of the dotted lines. Namely,

$$d(P_1F_2) - d(P_1F_1) = d(P_2F_1) - d(P_2F_2)$$

The vertices of the hyperbola are a distance a from the center, as shown below:



Looking at the point (the vertex) a units away from the center, we see that the distance from it to F_1 is $a+c$, where c is the distance from an center to a focus. In addition, we know that the distance from the same point to the other vertex is $c-a$. Taking the difference, we get

$$(c+a)-(c-a) = 2a$$

Thus, because the difference in distances between the foci of every point on the hyperbola is constant, we always know that the difference is $2a$. Referring back to the figure with the dotted and solid lines, this means that

$$d(P_2F_2) - d(P_2F_1) = 2a$$

Which means, if the point P_2 has coordinates (x,y),

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

Moving terms,

$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Now, our goal becomes to get rid of the radicals, just as we did when we derived the ellipse equation:

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Multiplying out,

$$x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+x^2-2xc+c^2+y^2$$

Cancelling and rearranging,

$$4cx-4a^2=4a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=a\sqrt{(x-c)^2+y^2}$$

Squaring again to get eliminate the square root,

$$(cx-a^2)^2=(a\sqrt{(x-c)^2+y^2})^2$$

$$c^2x^2-2a^2cx+a^4=a^2((x-c)^2+y^2)$$

Foiling,

$$c^2x^2-2a^2cx+a^4=a^2(x^2-2xc+c^2+y^2)$$

$$c^2x^2-2a^2cx+a^4=a^2x^2-a^22xc+a^2c^2+a^2y^2)$$

Rearranging,

$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$x^2(c^2-a^2)-a^2y^2=a^2(c^2-a^2)$$

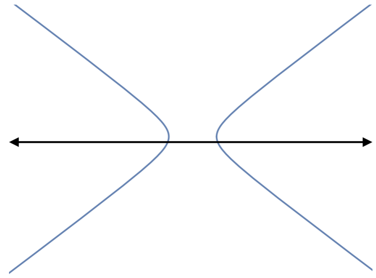
Now, we will let $b^2=c^2-a^2$

$$x^2b^2-a^2y^2=a^2b^2$$

Dividing all sides by a^2b^2, we get our final equation for a hyperbola centered at the origin (keep in mind that if we had a center of (h,k), then the x and y-containing terms would be $x-h$ and $(y-k)$, respectively:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{64}$$

This equation only works for a hyperbola with a horizontal transverse axis, which is the line of symmetry that runs through the hyperbola. In the image below, the black line is the transverse axis.



If we wanted to represent a hyperbola with a vertical transverse axis, the equation would be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{65}$$

Additionally, since we know that $b^2 = c^2 - a^2$, we know that c, or the distance from the center to the foci, is $\sqrt{a^2 + b^2}$

$$c = \sqrt{a^2 + b^2} \tag{66}$$

We can derive an equation for the asymptotes of a hyperbola as well. This is done simply by finding an equation for y:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Multiplying by a^2b^2,

$$b^2x^2 - a^2y^2 = a^2b^2$$

Subtracting b^2x^2,

$$-a^2y^2 = a^2b^2 - b^2x^2$$

Dividing by $-a^2$,

$$y^2 = -b^2 + \frac{b^2}{x^2}$$

Factoring out $\frac{b^2}{a^2}$,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

Now, we can solve for y by taking the square root of both sides.

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

As x gets bigger, the ratio of $\frac{x^2}{a^2}$ gets infinitely large. Because of this, we can eliminate the a^2 in the asymptote, because it becomes irrelevant as x increases. We can rewrite the expression as

$$y = \pm\frac{b}{a}\sqrt{x^2} = \pm\frac{b}{a}x$$

Obviously, with a hyperbola with center (h,k), the asymptotes are

$$\textbf{Horizontal transverse axis} \qquad y-k = \pm\frac{b}{a}(x-h) \tag{67}$$

If our hyperbola had a vertical transverse axis, the asymptotes would be

$$\textbf{Vertical transverse axis} \qquad y-k = \pm\frac{a}{b}(x-h) \tag{68}$$

The following table summarizes the properties of our two types of hyperbolas:

	Horizontal Transverse Axis	Vertical Transverse Axis
Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
c	$\sqrt{a^2+b^2}$	$\sqrt{a^2+b^2}$
Asymptotes	$y-k=\pm\frac{b}{a}(x-h)$	$y-k=\pm\frac{a}{b}(x-h)$

8.4 Eccentricity

The eccentricity of a conic section describes the type of conic section that it is. The definition of eccentricity is $e = \frac{c}{a}$, where c represents the distance from a focus to the center of the conic section and a represents the distance from a vertex to the center of the conic. Obviously, in the case of an ellipse, c will always be less than a because the focus is inside the ellipse, while a is the furthest point on the ellipse from the center. Because of this, the eccentricity of an ellipse, $e_{ellipse}$, will always be less than 1. In addition, since the focus of the circle is the center (namely the distance between the focus and center is zero), $e_{circle} = 0$, because $c = 0$. For a hyperbola, $e_{hyperbola} > 1$, since the focus of a hyperbola is farther from the center than the a vertex. Eccentricity is useful because it allows us to identify what type of conic we are dealing with, without actually having to graph it.

Example 8.3

Using eccentricity, dentify which type of conic section the following equation is:

$$9x^2 - 90x + 12y - y^2 = -81$$

First, we should get the equation into a form that we can work with. Namely,

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1$$

In order to do this, we have to complete the square.

$$9(x^2 - 10x + 25) - (y^2 - 12y + 36) = -81 - 36 + 9(25)$$

Now, we can get an $(x-h)^2$ and $(y-k)^2$:

$$9(x-5)^2 - (y-6)^2 = 108$$

Dividing by 108 to get our final form,

$$\frac{(x-5)^2}{12} - \frac{(y-6)^2}{108} = 1$$

Right off the bat, we can tell $a = \sqrt{12}$ and $b = \sqrt{108}$. Since $c = \sqrt{a^2 + b^2}$, we can see that $c = \sqrt{120}$. Now, we can determine the eccentricity:

$$e = \frac{c}{a} = \frac{\sqrt{120}}{\sqrt{12}} = \sqrt{10}$$

Since $\sqrt{10} > 1$, we know that the equation $9x^2 - 90x + 12y - y^2 = -81$ is a hyperbola.

8.5 Parametric Equations

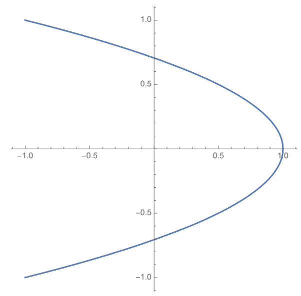
This section will provide a more in-depth analysis of parametric equations. As said before, an ellipse with center (h, k) can be described by the parametric equations

$$x(t) = R_x \cos(\theta(t)) + h$$
$$y(t) = R_y \sin(\theta(t)) + k$$

If $R_x = R_y$, then the graph is obviously a circle; it is just an ellipse with a constant radius (equal values of R_x and R_y). Because of this, $c = 0$, which means the eccentricity is 0.
If we were to change the coefficient of one of the arguments, we would generate a parabola. For example, consider the following set of parametric equations:

$$x(t) = \cos(2(\theta(t)))$$
$$y(t) = \sin(\theta(t))$$

If we compare $x(t)$ to $y(t)$, we can see that the period of $x(t)$ is half of $y(t)$, due to the coefficient. This results in a parabolic shape. However, it is not a complete parabola, because of the ranges of the sine and cosine functions:



Without any transformations, the domain and range of our parabola is $[-1,1]$.
Hyperbolas can be described by the parametric equations

$$\begin{aligned} x(t) &= a\sec(\theta(t))+h \\ y(t) &= b\tan(\theta(t))+k \end{aligned}$$ if the hyperbola has a **horizontal** transverse axis

$$\begin{aligned} x(t) &= b\tan(\theta(t))+h \\ y(t) &= a\sec(\theta(t))+k \end{aligned}$$ if the hyperbola has a **vertical** transverse axis

For a circle with radius r and center (h,k), the parametric equations are

$$\begin{aligned} x(t) &= r\cos(\theta(t))+h \\ y(t) &= r\sin(\theta(t))+k \end{aligned}$$

An important thing to note about these equations is that the trigonometric operators do not always have to be cosine for $x(t)$ and sine for $y(t)$. It all depends on where the graph is being started. For example, consider a common application of parametric equations, a problem involving a ferris wheel.

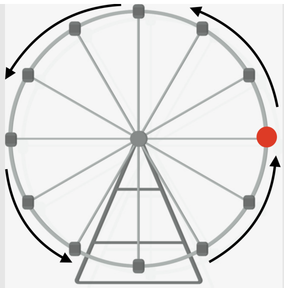
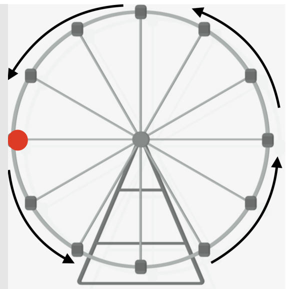
Example 8.4

Consider a ferris wheel with radius 20ft and the center 12ft off the ground. If the time it takes to make one complete revolution is 15s and the rider starts from the bottom, going counterclockwise,

(a) Find a set of parametric equations for the rider if the they start at the bottom of the wheel.

(b) Where is the rider after 6s?

First off, we know that the period, T, of the ferris wheel is 15 seconds. Because of this, the frequency is $\frac{2\pi}{\text{period}}$, or $\frac{2\pi}{15s}$. In addition, the center of the ferris wheel is $(0,12)$. To find what trigonometric operators we are using for the $x(t)$ and $y(t)$ functions, we must consider the path the rider takes. Looking strictly at their x position relative to time, it increases for $\frac{15}{4}$ seconds, or $\frac{1}{4}T$, and then decreases. At $t=\frac{3}{4}T$, the rider is at a minimum x component.

$t = \frac{1}{4}P$ $t = \frac{3}{4}P$

Looking at these two points, we see that the x is increasing for a quarter of the period, decreasing for the next half, and then increasing for the remaining quarter. This is the same behavior of the sine function as θ increases. Because of this, the sine operator will be the trigonometric operator of our choice for the $x(t)$ equation. Finally, the amplitude, or radius of our x is 10.

$$x(t) = 10\sin(\frac{2\pi}{15}t)$$

Considering the y as a function of time, it starts at a minimum and reaches a maximum height at time $t = \frac{1}{2}P$. Because of this, the y-component behaves like a negative cosine function. With a center at $(0, 12)$, we know that our k value is 12. Therefore,

$$y(t) = -10\cos(\frac{2\pi}{15}t) + 12$$

Our set of parametric equations are

$$\boxed{\begin{array}{c} x(t) = 10\sin(\frac{2\pi}{15}t) \\ y(t) = -10\cos(\frac{2\pi}{15}t) + 12 \end{array}}$$

b) At time $t = 6s$,

$$x(6) = 10\sin(\frac{2\pi}{15}6) = 5.878$$

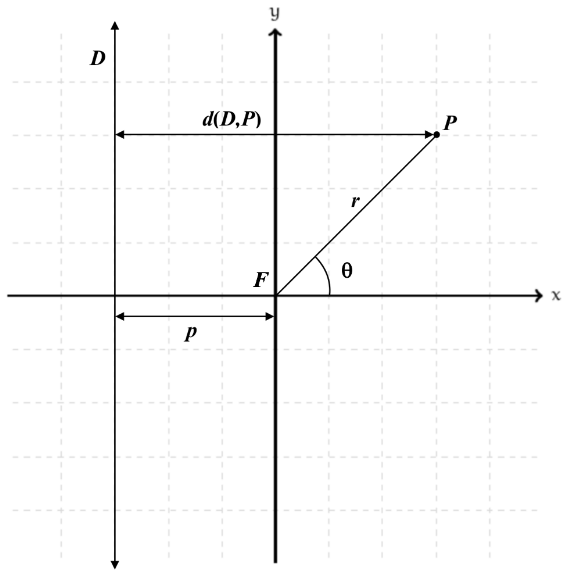
$$y(6) = -10\cos(\frac{2\pi}{15}6) + 12 = 20.090$$

Therefore, the rider can be represented by the coordinates $\boxed{(5.878, 20.090)}$ at time $t = 6s$.

Just a note: we could have elliptical "ferris wheels," in which we would do apply the similar concepts.

8.6 Conics in Polar Form

The equation of an conics in polar form is not intuitive (besides circles), given that it is hard to visualize how r varies directly with θ. However, we can prove the formula using the concept of eccentricity and the properties of our rectangular equations that we already proved. We know that the distance from a focus to a point divided by the distance from the directrix to the point is the eccentricity; this is also known as $\frac{c}{a}$. Consider the diagram below of an arbitrary polar point P with focus F and directrix D, where p is the length from the directrix to the origin.



As said before, we know

$$\frac{d(F,P)}{d(D,P)} = e$$

Multiplying through,

$$d(F,P) = ed(D,P)$$

In addition, we can see from the diagram that

$$d(D,P) = p + r\cos\theta$$

We are simply projecting r onto the x-axis.
Additionally, we know that the distance from the focus to a point, or $d(F,P)$, is r:

$$d(F,P) = r$$

Substituting these values into $d(F,P) = ed(D,P)$,

$$r = e(p + r\cos\theta) = ep + er\cos\theta$$

Solving for r, we get

$$r(\theta) = \frac{ep}{1 - e\cos\theta} \tag{69}$$

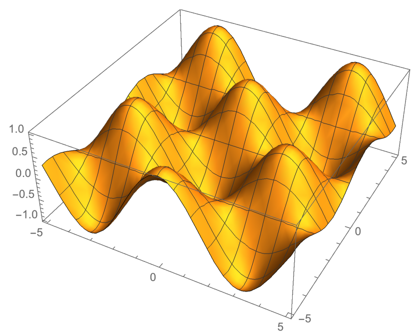
This equation is only true for a conic with a vertical directrix to the left the pole, as shown in the diagram. If we were to solve for $r(\theta)$ for the three remaining cases, we would get the remaining formulas. The different resulting polar conic equations are shown in the table below:

$r(\theta) = \frac{ep}{1-e\cos\theta}$	Vertical directrix at a distance p units to the left of the pole
$r(\theta) = \frac{ep}{1-e\cos\theta}$	Vertical directrix at a distance p units to the right of the pole
$r(\theta) = \frac{ep}{1-e\sin\theta}$	Horizontal directrix at a distance p units below the pole
$r(\theta) = \frac{ep}{1+e\sin\theta}$	Horizontal directrix at a distance p units above the pole

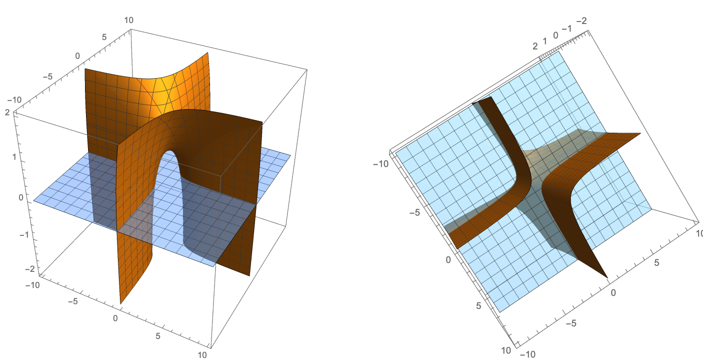
One useful thing to note about these formulas is that they apply to **all** conics; not just an individual conic as, for example, the ellipse equation.

8.7 3-D Surfaces

This chapter will introduce a new variable in our equations, z. This allows us to have three dimensional graphs, where a surface is graphed, rather than a line. For example, consider the graph of $z = \cos x \sin y$:



Obviously, if we were given a surface, it would be much harder to come up with an equation than if we were given a curve. However, there are some general 3-D surface shape equations that are worth knowing. We classify a lot of these shapes by describing their plane traces. A plane trace is the 2-dimensional curve formed by the intersection of a plane with the surface. The three main traces we use will be the xy trace, the yz trace, and the xz trace. To find the equations of each trace, we simply set the other variable not in the plane to zero. For example, if we wanted to find the xy trace of the equation $xy + e^z = 2$, we would set z to 0. The resulting equation for our xy trace would be $xy + 1 = 2$ or $xy = 1$, which happens to be the parent rational function. This is shown in the figure below, where the translucent blue plane is the xy plane.

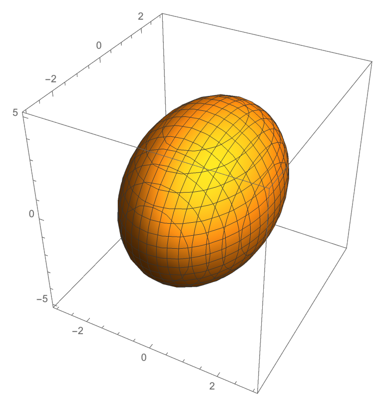


Ellipsoids

An ellipsoid centered around the point (x_0, y_0, z_0) has the equation

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

This is clearly similar to the equation of a normal ellipse, except this time we have a $\frac{z^2}{c^2}$ term. All traces of an ellipsoid are ellipses. Consider the graph of an ellipsoid, $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$



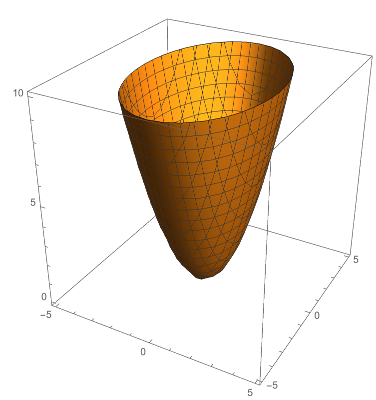
The values of *a, b*, and *c* determine each radius R_x, R_y, and R_z. In addition, if $R_x = R_y = R_z$, namely $a = b = c$, then the ellipsoid is a sphere.

Elliptic Paraboloids

An elliptic paraboloid has the general equation

$$\frac{z-z_0}{c}=\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}$$

The xy trace is an ellipse, while both vertical traces are parabolas. Consider the graph of $\frac{z}{4} = \frac{x^2}{4}+\frac{y^2}{9}$:



The equation for an elliptic paraboloid can be better thought of as a 3-dimensional quadratic function; namely,

$$z = ax^2 + by^2$$

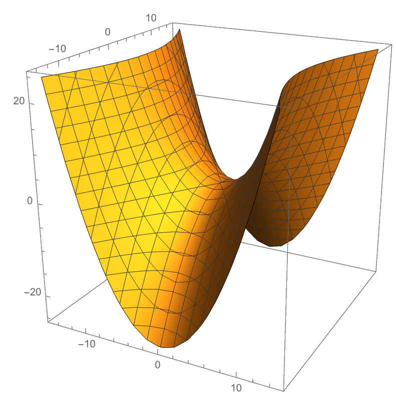
This equation still represents the same general shape, since a, b, and c are just arbitrary constants.

Hyperbolic Paraboloids

A hyperbolic paraboloid is essentially the same as an elliptical paraboloid, except its xy traces are hyperbolas. The vertical traces, however, are still parabolas. The equation is similar to an elliptical paraboloid as well, except we subtract a term rather than adding:

$$\frac{z - z_0}{c} = \frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2}$$

Consider the graph of a hyperbolic paraboloid, $z = \frac{x^2}{4} - \frac{y^2}{9}$:

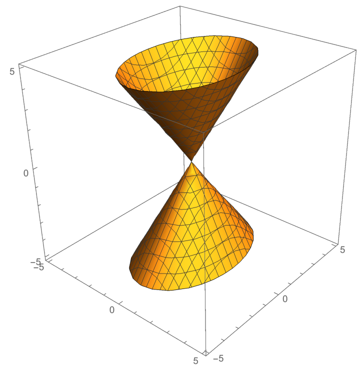


Cones

A cone has the general equation

$$\frac{(z - z_0)^2}{c^2} = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}$$

This is if we want the cone to open up on the z-axis. The xy trace of a cone is a degenerate conic. However, if we moved our horizontal trace slightly off the xy plane, we would get an ellipse. The yz and xz traces are intersecting lines, but if moved slightly from the center as well, are hyperbolas. If we wanted the cone to open up on the x or y-axis, then the equations would be $\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$ and $\frac{y^2}{b^2} = \frac{x^2}{a^2} + \frac{z^2}{c^2}$, respectively.
Consider the graph of $\frac{z^2}{16} = \frac{x^2}{4} + \frac{y^2}{9}$:



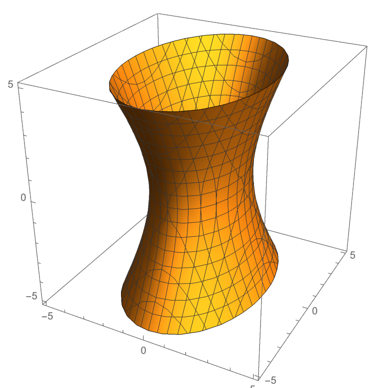
If we wanted the cone to be circularly symmetric about the z-axis, then we would set $a = b$, assuming the cone is centered around the origin.

Hyperboloid of One Sheet

A hyperboloid of one sheet has the general equation

$$\frac{(z-z_0)^2}{c^2}+1=\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}$$

This is strikingly similar to the equation of a cone, which had the equation $\frac{(z-z_0)^2}{c^2} = \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$. However, we simply added 1 to the side with z. The resulting graph looks similar to a cone, but with an elliptical xy trace for all its cross sections. The cone always has one point where its trace will be a degenerate conic. Take a look at the one-sheet hyperbolid, $1+\frac{z^2}{16}=\frac{x^2}{4}+\frac{y^2}{9}$

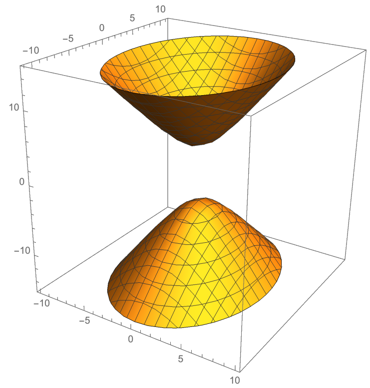


Hyperboloid of Two Sheets

A hyperboloid of two sheets is represented by almost the same equation as a hyperboloid of one sheet. However, the only difference is that we subtract 1:

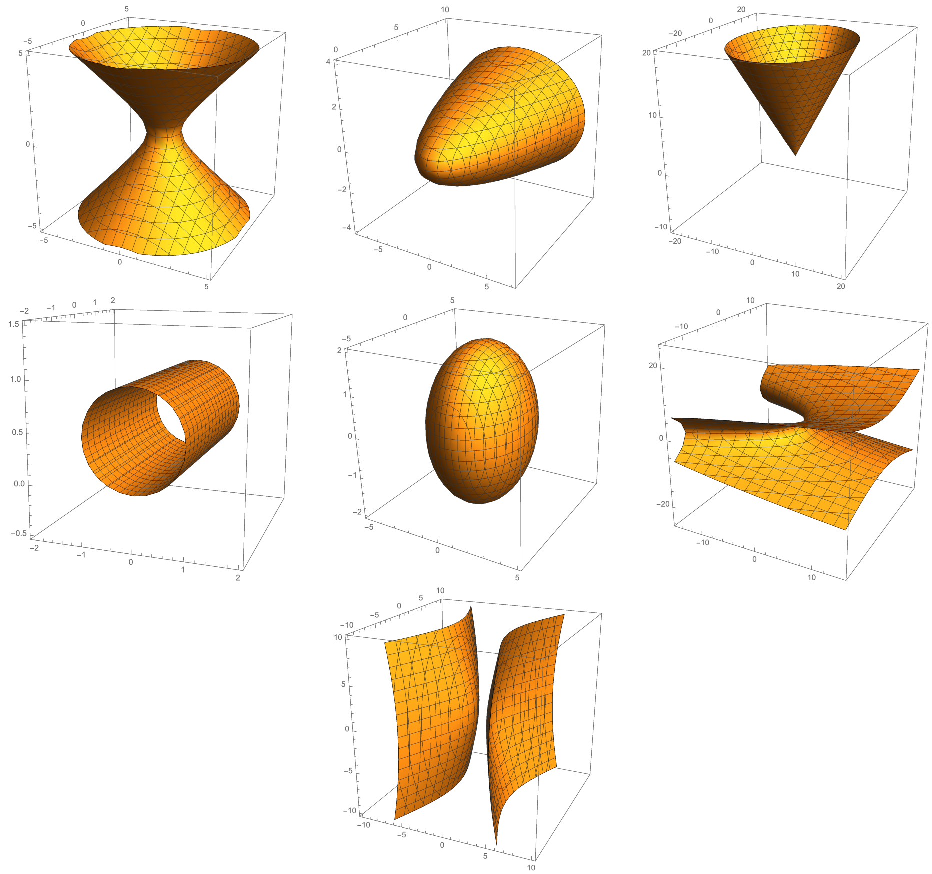
$$\frac{(z-z_0)^2}{c^2}-1=\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}$$

As you have probably guessed by now, the shape looks similar to a cone, but with its two "parts," separated:



8.8 Problems

1. Graph the following conic sections
   (a) $8y-6x+16+x^2+y^2=0$
   (b) $y^2+6y+8x-23=0$
   (c) $y^2-x^2=4$
   (d) $y^2+2y+12x+25=0$
   (e) $2y^2-3x^2-4y+12x+8=0$
   (f) $\frac{(y+3)^2}{9}-\frac{(x-1)^2}{4}=1$
2. Graph the following polar conic sections. In addition, write the section in rectangular form.
   (a) $r(\theta)=\frac{3}{2+2\sin\theta}$
   (b) $r(\theta)=\frac{2}{1-\cos\theta}$
   (c) $r(\theta)=(\frac{1}{\cos^2\theta}-1)\cot^2\theta\frac{1}{1-3\sin\theta}$
   (d) $r(\theta)=\frac{3}{1-2\cos\theta}$
3. Find an equation for a hyperbola with
   (a) vertices $(0,\pm 2)$ and asymptotes $y=\pm 2x$
   (b) vertices $(0,\pm 2)$ and foci $(0,\pm 3)$
   (c) point $(2,4)$ and asymptotes $y=\pm x$
4. A tilted cylinder has equation $(x^2-2y-2z)^2+(y-2x-2z)^2=1$. Show that the water surface at $z=0$ is an ellipse. What is its equation?
5. Match the following 3-D surfaces with their equations
   (a) $\frac{z^2}{4}-\frac{y^2}{9}=x^2$
   (b) $\frac{x^2}{4}+z^2=z$
   (c) $x^2+y^2=z^2+1$
   (d) $\frac{x^2}{4}+z^2=y^2$
   (e) $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{4}=1$
   (f) $\frac{z^2}{9}+\frac{y^2}{4}=x^2-1$
   (g) $\frac{z}{3}=\sqrt{\frac{x^2}{4}+\frac{y^2}{16}}$



6. Prove that the xy trace of $z = \cos x \sin y$ is a grid.

7. An elliptical theater is to be constructed such that the stage is at one focus and a recording device is at the other focus. The backstage area must be 10 feet at its deepest point. The two exits are at the co-vertices and are 80 feet apart. How far is the recording device from the sage?

8. An amusement park has an elliptical ferris wheel with its lowest point 4 feet off the ground and its highest point 32 feet off the ground. The width in the middle is 20 feet. It takes 22 seconds to make 1 revolution. Arya gets on the ride at the bottom and begins to move clockwise, and simultaneously with her start, her friend Jon throws a ball to her. Jon is standing on the ground 25 feet to the left of the starting point of the ferris wheel. He releases the ball with an initial velocity of $40\frac{\text{ft}}{\text{s}}$ at an angle $74°$ above the horizontal.

   (a) Write the parametric equations for the ferris wheel and the ball

   (b) How far does Arya have to reach out to catch the ball?

   (c) If instead Jon throws the ball 14 seconds after Arya begins the ride, how long after Arya begins the ride does she come closest to the ball?

9. A narrow arch supporting a stone bridge is in the shape of half an ellipse and is 12 meters long and 3 meters high. A person standing at one focus of the ellipse throws a rubber ball against the arch. No matter what direction the ball is thrown, it always bounces off the arch once and hits the same point on the ground. How far apart are the person throwing the ball and the point on the ground at which the ball strikes?

10. Bronn is standing on the ground, 20 feet to the left of the center of a rose petal ferris wheel. The ferris wheel takes 32 seconds to make one revolution and has the equation $r(\theta) = 10\cos(4\theta(t) + \phi)$, where $\theta(t)$ represents the angle θ as function of time. Tyrion rides the ferris wheel, starting from the bottom. Neglect acceleration due to gravity.

    (a) What is ϕ, given that Tyrion is at the bottom when $t = 0$?

    (b) Bronn throws a baseball at an angle 72° off the ground, hoping for Tyrion to catch it at the top of the ferris wheel. (Use projectile motion equations)

        i. What speed must he throw the ball at?

        ii. When should he release the ball if he wants Tyrion to catch it at the top?

11. Three tracking stations, A, B, and C, are represented by the following coordinates (each unit is one kilometer):

    A: $(0, 100)$
    B: $(0, 0)$
    C: $(50, 0)$

    When a ship emitted a signal, it travelled 60km farther to reach B than to reach A, and travelled 20km farther to reach B than C. Find the coordinates of the ship.

9 Sequences and Series

This chapter will cover and expand on sequences and series. Series, in particular, are extremely useful for a number of things. For example, if we wish to find the sum of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$, we could just keep adding terms until we get within some epsilon to the actual value. However, there is a much quicker, not-so-obvious way to go about the problem. Throughout this chapter, we will derive a number of finite-termed formulas to simplify infinite sums, using them to solve real-life applications.

9.1 Arithmetic and Geometric

First off, we need to make sure we know the difference between an arithmetic and geometric sequence. An arithmetic sequence is defined as a sequence where each consecutive term differs from the previous term **by a constant difference**. This constant difference between terms is the same for all of terms in the sequence. Examples of arithmetic sequences include

$$1,3,5,7,9,11...$$
$$9,5,1,-3,-7...$$
$$1,100,199,298...$$

Where the constant differences are $2,-4$, and 99, respectively.
A geometric sequence is defined as a sequence where each consecutive term differs from the previous term **by a constant ratio**. This constant ratio between terms is the same for all of terms in the sequence. If the ratio r has a magnitude $|r|\geq 1$, the sequence **diverges**. If $|r|<1$, the sequence **converges**. Convergence and divergence will be discussed later. Examples of geometric sequences include

$$2,4,8,16,32,64...$$
$$3,-9,27,-81,243...$$
$$64,16,4,1,\tfrac{1}{4}...$$

Where the ratios r are $2,-3$, and $\frac{1}{4}$, respectively. The first two sequences diverge, while the last one converges.

9.2 Sequences

We define a sequence as a function with a domain of positive integers (1,2,3...). We call this function a_n or $a(n)$. Sequences can either be *explicitly defined*, meaning that each term is a function directly of n, or *recursively defined*, meaning that each term is dependent on the value of the previous term. We denote a recursive sequence as $a(n)=f(a_{n-1})$. The $f(a_{n-1})$ means a function of a_{n-1}; this could be $2a_{n-1}$, $(a_{n-1})^2$, $\ln(a_{n-1})$, etc.

Example 9.1
Find $a(n)$ and $f(a_{n-1})$ for the following geometric sequence:

$$8,32,128,512...$$

Clearly, by looking at the sequence, it is not linear. However, by looking at the pattern, we can see that each successive number is 4 times the previous one. Because of this, $a(n)$ is going to be exponential with a base of 4. So, we know that $a(n) = C(4^n)$, where C is some constant. To figure out C, we can plug in for a_1:

$$a(1) = C(4) = 8$$

$$C = 2$$

Plugging back into a_n, we get

$$\boxed{a_n = 2(4^n)}$$

To find $f(a_{n-1})$, we know that each term is four times the previous term. Thus,

$$a_n = 4a_{n-1} \quad \text{where } a_1 = 8$$

There is not one easy trick to find a_n for a given sequence; it is, like many aspects of mathematics, something that lots of practice to get the "feel" for it.

Convergence and Divergence

We say a sequence *converges* if the limit as n approaches infinity of a sequence is a defined finite number. If the sequence does not meet this condition, then it *diverges*. Namely,

$$\text{A sequence converges if } \lim_{n\to\infty} a_n = L, \text{where } L \in \mathbb{R} \tag{70}$$

Determine whether the following sequences diverge or not:[8]

1. $a_n = \frac{n+2}{n^2}$ 2. $a_n = \frac{n^3+3n^2}{3n^3-4n^2}$ 3. $a_n = \cos n$ 4. $a_n = (-1)^n \cos(\pi n)$ 5. $a_n = \sin(\frac{1}{n})$

9.3 Series

A series best defined as *the sum of a sequence*. We express a sum using sigma notation - the weird "E" thing you always saw growing up but never knew what it actually meant. We define the sum of a sequence from a_1 to a_n as

$$\sum_{i=1}^{k} a(i) = a_1 + a_2 + a_3 + a_4 + ... + a_k$$

If $k \neq \infty$, then we call the sum a *partial sum*. If $k = \infty$, we call the sum an *infinite sum*.

[8] 1. Converges 2. Converges 3. Diverges 4. Converges 5. Converges

Example 9.2

Find the partial sum from $i = 1$ to $i = 5$ of the sequence $a_n = 2^n$

We can write the sum as $\sum_{i=1}^{5} a_i$, or $a_1 + a_2 + a_3 + a_4 + a_5$. Evaluating the sum, we get $2^1 + 2^2 + 2^3 + 2^4 + 2^5 = \boxed{62}$

There are also several properties of sums that we can prove. To start off, consider the sums $\sum_{i=1}^{k} a_n$ and $\sum_{i=1}^{k} Ca_n$, where C is a constant.

We can rewrite $\sum_{i=1}^{k} a_n$ as $a_1 + a_2 + a_3 + ... + a_k$. Additionally, we can rewrite $\sum_{i=1}^{k} Ca_n$ as $Ca_1 + Ca_2 + Ca_3 + ... + Ca_k$. We can factor a C out of this equation:

$$Ca_1 + Ca_2 + Ca_3 + ... + Ca_k = C(a_1 + a_2 + a_3 + ... + a_k)$$

However, there is something important to notice about this equation:

$$C(a_1 + a_2 + a_3 + ... + a_k) = C\sum_{i=1}^{k} a_i$$

Namely, we can pull a constant out of a sum:

$$\sum_{i=1}^{k} Ca_i = C\sum_{i=1}^{k} a_i \tag{71}$$

Additionally, consider the sum $\sum_{i=1}^{k} a_n + b_n$. When we expand this sum, we get

$$a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + ... + a_k + b_k$$

We can rewrite this sum, however, as

$$(a_1 + a_2 + a_3 + ... + a_k) + (b_1 + b_2 + b_3 + ... + b_k)$$

which can be put into sigma notation as $\sum_{i=1}^{k} a_n + \sum_{i=1}^{k} b_n$. Therefore,

$$\sum_{i=1}^{k} a_n + b_n = \sum_{i=1}^{k} a_n + \sum_{i=1}^{k} b_n \tag{72}$$

Similarly,

$$\sum_{i=1}^{k} a_n - b_n = \sum_{i=1}^{k} a_n - \sum_{i=1}^{k} b_n \tag{73}$$

9.4 Series Summation Formulas

In some specific types of series, we have formulas that allow us to easily compute the sum without adding each individual term of a_n. First, consider the sum $\sum_{i=1}^{n} 1$. This sequence, written out, would be $1+1+1+1+\ldots$ We would add 1 n times. Because of this, the sum of $\sum_{i=1}^{n} 1$ is equivalent to n.

$$\sum_{i=1}^{n} 1 = n \tag{74}$$

Example 9.3

Evaluate the sum $\sum_{i=1}^{14} 3$

We can take out the constant 3 to get

$$3\sum_{i=1}^{14} 1$$

Evaluating the sum, we get

$$3(14) = \boxed{42}$$

Now, consider the sum $S_n = \sum_{i=1}^{n} i$. Expanding, we get

$$1+2+3+4+5+\ldots+n = S_n$$

Rewriting this sum backwards,

$$n+(n-1)+(n-2)+(n-3)+\ldots+4+3+2+1$$

Now, if we want to add another S_n to the sequence,

$$\begin{array}{ccccccccccccc} & n & + & n-1 & + & n-2 & +\ldots+ & 3 & + & 2 & + & 1 \\ + & 1 & + & 2 & + & 3 & +\ldots+ & n-2 & + & n-1 & + & n \\ \hline & n+1 & + & n+1 & + & n+1 & +\ldots+ & n+1 & + & n+1 & + & n+1 \end{array}$$

The important thing to notice here is that this resulting sum is equal to $2S_n$, or $n(n+1)$. This is because we are adding $(n+1)$, n times, because there are n terms in the sequence. Therefore,

$$2S_n = n(n+1)$$

Which means

$$S_n = \frac{n(n+1)}{2}$$

Therefore,

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{75}$$

Example 9.4

Find the sum $S_n = \sum_{i=1}^{12}(3n-6)$

We can rewrite the S_n as $\sum_{i=1}^{12} 3n - \sum_{i=1}^{12} 6$. Pulling constants out, we get

$$3\sum_{i=1}^{12} n - 6\sum_{i=1}^{12} 1$$

Now, we can use our summation formulas to get

$$3\frac{(13)(12)}{2} - 6(12)$$

$$= \boxed{162}$$

Moving on, consider the sum $\sum_{i=1}^{n} i^2$. To prove this identity, we will use mathematical induction (see page 154 if this is not familiar to you). The claim is that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$;
Starting off with *n=1*,

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6} = 1$$

Obviously, $\sum_{i=1}^{1} i^2 = 1$, so the first case checks out. Now, we will assume that the statement is true for $n = k$, where $k > 1$. Namely,

$$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now, letting $n = k+1$, we must prove

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Since we know $1^2+2^2+3^2+...+k^2 = \frac{k(k+1)(2k+1)}{6}$ based off our original assumption, we can substitute.

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Now, we have to prove this equality by brute force. Simplifying,

$$k(2k+1)+6(k+1) = (k+2)(2k+3)$$

$$2k^2+7k+6 = 2k^2+7k+6$$

Through mathematical induction, we have proved that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{76}$$

For all $n \geq 1$ Lastly, we will prove the sum $\sum_{i=1}^{n} i^3$. As done before, we will use induction to prove our claim that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Letting $n=1$,

$$\left(\frac{1(1+1)}{2}\right)^2 = 1$$

Assuming that the statement holds true for $n=k$,

$$1^3+2^3+3^3+...+k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Letting $n=k+1$, we will prove the following statement is true based off our assumption with $n=k$:

$$1^3+2^3+3^3+...+k^3+(k+1)^3 = \left(\frac{(k+1)[(k+1)+1]}{2}\right)^2$$

$$\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$\frac{k^2(k+1)^2}{4}+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{k^2}{4}+(k+1) = \frac{(k+2)^2}{4}$$

$$k^2+4k+4 = k^2+4k+4$$

Thus, our statement

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{77}$$

is true for $n \geq 1$.

As you have probably learned by now, all of our summation formulas are dependent on the fact that our sum starts from $i = 1$. How would we, for example, evaluate the sum of sequence a_n when the sum does not start from $n = 1$, but rather $n = a$, where $a > 1$?

$$S = \sum_{n=b}^{c} a_n$$

This sum, expanded, would be

$$S = a_b + a_{b+1} + a_{b+2} + ... + a_c$$

The claim here is that we can rewrite the sum $S = \sum\limits_{n=b}^{c} a_n$ as a difference of sums:

$$S = \sum_{n=b}^{c} a_n = \sum_{n=1}^{c} a_n - \sum_{n=1}^{b-1} a_n \tag{78}$$

Expanding the equation,

$$a_b + a_{b+1} + a_{b+2} + ... + a_c = (a_1 + a_2 + ... + a_{b-1} + a_b + ... + a_c) - (a_1 + a_2 + ... + a_{b-1})$$

Now, notice that the terms $(a_1 + a_2 + ... + a_{b-1})$ cancel out on the right side. Once we do this, we prove equation (78):

$$a_b + a_{b+1} + a_{b+2} + ... + a_c = a_b + ... + a_c$$

Summarizing the main summation formulas for this section,

$\sum\limits_{i=1}^{n} 1 = n$
$\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$
$\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum\limits_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
$\sum\limits_{n=b}^{c} a_n = \sum\limits_{n=1}^{c} a_n - \sum\limits_{n=1}^{b-1} a_n$

9.5 Geometric Sequences and Series

A geometric sequence has the form $a_n = a_1 r^{n-1}$. Clearly, each term in the sequence is the previous term multiplied by a constant ratio. There are numerous applications of geometric series in real life; consider the following example.

Example 9.5

A bouncy ball is dropped from height h, and each is a constant multiplied by the previous height; $h_{n+1} = \alpha h_n$, where $0 < \alpha < 1$.

(a) Find a geometric sequence that represents the height a ball bounces after n bounces.

(b) Find a formula that includes a geometric series (in sigma notation) that represents the total distance the ball has traveled by the n'th time it hits the ground.

First off, we know that our ratio r is α, because each consecutive bounce is α the height of the previous bounce height. Because of this, we can follow the general geometric sequence form of $a_n = a_1 r^{n-1}$. Since a_1, or the height after the first bounce, is αh, our sequence is $h_n = \alpha h(\alpha)^{n-1}$. This simplifies to

$$\boxed{h_n = h(\alpha)^{n-1}}$$

If we want to find a geometric sequence that represents the total distance traveled by the ball, we must conceptually think about the motion of the ball before we construct any meaningful formula. First off, for $n = 1$ bounces, the ball has simply been dropped from height h and hit the ground. After $n = 2$ bounces, the ball has been dropped, bounced up a height αh, and fallen back a distance αh. The total distance traveled is $h + 2\alpha h$. After the second bounce, it goes up a height $\alpha(\alpha h)$, or $\alpha^2 h$. Thus, for $n = 3$ bounces, the total distance is $h + 2\alpha h + 2\alpha^2 h$... the pattern continues. So, by the n'th time the ball bounces, we have a total distance $h + 2\alpha h + 2\alpha^2 h + ... + 2\alpha^n h$. Because of this, we can represent the total distance traveled by the time the ball hits the ground the n'th time as

$$S_n = h + 2\sum_{k=1}^{n} \alpha^n h$$

As stated before, we know that if the geometric sum has a value of $|r| < 1$, it converges to a specific value. If we want to find the specific sum of an infinite geometric series $\sum_{i=1}^{n} a_1 r^{i-1}$, we must take the limit as n approaches infinity. Namely,

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1}$$

Expanding the sum, we get

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots$$

We can factor out an r to get

$$S = r(a_1 r^{-1} + a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots)$$

Notice that we can substitute S in on the right side of the equation:

$$S = r(a_1 r^{-1} + S)$$

$$S = a_1 + rS$$

Now that we have an equation of finite terms for S, we just have to solve for it! Rearranging and factoring out an S,

$$S(1-r) = a_1$$

Thus, we have our formula for the sum of an infinite converging geometric series.

$$S = \frac{a_1}{1-r} \tag{79}$$

If we wish to find the sum of a geometric series of n terms, we can derive a formula in a somewhat similar manner:

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots a_1 r^{n-1}$$

Multiplying both sides by r,

$$rS_n = r(a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots a_1 r^{n-1})$$

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots a_1 r^n$$

Now, we know that the right side is $S_n + a_1 r^n - a_1$; look at the first equation!

$$rS_n = S_n + a_1 r^n - a_1$$

Rearranging and factoring out S_n

$$S_n - rS_n = a_1 - a_1 r^n$$

$$S_n(1-r) = a_1(1-r^n)$$

Alas, we have an explicit formula to represent the sum of a converging geometric sequence with n terms:

$$S_n = a_1 \frac{1-r^n}{1-r} \tag{80}$$

The infinite sum formula can be proved from this as well; taking the limit of S_n as n approaches infinity,

$$\lim_{n\to\infty} a_1 \frac{1-r^n}{1-r}$$

Since $|r| < 1$, we know that if we multiply r by itself an infinite amount of times, it will be zero. Namely,

$$\lim_{n \to \infty} r^n = 0$$

Thus,

$$S_\infty = \frac{a_1}{1-r}$$

The main takeaway from this is that the infinite sum of a converging geometric series really is a limit. Although we have an explicit formula for the infinite sum, it just happens that the n-containing term in the series **always** converges to zero if $r < 1$.

9.6 Compound Interest

A common application of sequences and series is the compound interest model. The model represents an account with money in it. The money in the account may either increase decrease with time; for this model, we will assume it strictly increases. Given the interest rate r, the initial amount of money M, and the number of deposits made each year N, we can *recursively* define the amount of money in an account:

$$a_0 = M$$
$$a_n = (1 + \frac{r}{N})a_{n-1}$$

Example 9.6

An account has an initial balance of \$200. If it is compounded quarterly (four times per year) with an interest rate of 5%, find the amount in the account after two years.

We can start off simply by saying what variables we have.

$$r = 0.05$$
$$M = 200$$
$$N = 4$$

After two years, the account will have 8 deposits made; namely, we want to find a_8. Setting up our equation,

$$a_n = (1 + \frac{.05}{4})a_{n-1}$$

We can plug into our calculator by setting our mode to "seq" and solving for a_8

$$\boxed{a_8 = \$220.90}$$

9.7 Problems

1. Write general and recursive formulas for the following sequences.
   (a) $1, 4, 7, 10...$
   (b) $1, -4, 7, -10...$
   (c) $2, 5, 10, 17, 26...$
   (d) $1, 0, 1, 0, 1, 0, 1, 0...$
   (e) $1, 0, -1, 0, 1, 0, -1...$
   (f) $0, \frac{3}{9}, \frac{6}{16}, \frac{9}{25}, \frac{12}{36}...$
   (g) ★ Express with a trigonometric function: $1, -3, 5, -7, 9...$
2. Find the following sums
   (a) $\sum_{i=1}^{\infty} \left(\frac{3}{5}\right)^{n}$
   (b) $\sum_{i=1}^{\infty} \frac{3^{n-1}}{5^{n+2}}$
   (c) $\sum_{i=1}^{\infty} 4\left(-\frac{1}{3}\right)^{i-1}$
   (d) $\sum_{i=1}^{\infty} \left(\frac{2}{7}\right)\left(\frac{5^{i+1}}{3^{2i-1}}\right)$
3. State whether or not the following sequences converge or diverge. Justify your answers with a limit statement.
   (a) $a_n = \frac{1}{n}$
   (b) $a_n = \frac{2^n}{n!}$
   (c) $a_n = \frac{\sin n}{n}$
   (d) $a_n = \sin(\cos^{-1}(n))$
   (e) $a_n = \frac{n!}{n^n}$
4. State whether or not the following series converge or diverge. Justify your answers with a limit statement.
   (a) $\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{n}$
   (b) $\sum_{i=1}^{\infty} \left(\frac{5}{3}\right)^{n+3}$

(c) $\sum_{i=0}^{\infty}\left(\frac{4^{i/2}}{3^i}\right.$

(d) $\sum_{i=1}^{\infty}\left(\frac{9}{10}\right)^{1-n}$

5. Express the following sums explicitly in terms of n:

   (a) $\sum_{i=1}^{n} 3i^2+2i+6$

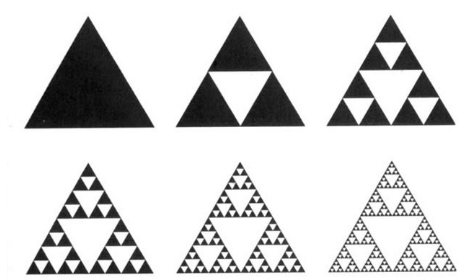
   (b) $\sum_{i=3}^{n}(1-i)^2+3i-5$

   (c) $\sum_{i=1}^{n} 2i^3+(2+2i)^2$

6. If $a_n = 3(2)^{n+1}$, for what value of n will $\sum_{i=1}^{n} a_n = 372$?

7. You are on a strange planet with your favorite bouncy ball. Curious as to how objects act on this strange surface, you decide to drop the ball from height h; strangely enough, the ball bounces even higher than before, to a height αh, where $\alpha > 1$. Then, something even stranger happens. After the k'th bounce, the gravity field strength decreases, allowing the bouncy ball to bounce a height βh_{prev}, where $0 < \beta < 1$ and h_{prev} is the height of the previous bounce. What is the total distance the ball bounces as the number of bounces n approaches infinity? Express your answer in terms of h, k, α, and β.

8. The Sierpinski triangle is a famous mathematical shape in that it is a fractal; it is formed by taking out a section of a triangle to form three new triangles, and then applying the same operation to each of the next three triangles, like so:

   If we let the first triangle be the $n = 1$ triangle and have an area of 1, find the area of all the successive triangles combined (the number of successive triangles is infinite). What is the area of the triangle as n approaches infinity? Prove using a limit statement.

9. Prove that $0.9999... = 1$

10. For what value of c is $\sum_{i=2}^{\infty}(1+c)^{-i} = \frac{9}{4}$?

11. $\left(\frac{1}{5}\right)^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)}$

12. ★ Prove the Arithmetic-Mean Geometric-Mean inequality:

$$\frac{x_1+x_2+x_3+\ldots+x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

10 Intro to Calculus: Limits

This chapter will provide a brief introduction to Calculus: the majority of the math that you will be doing after this course. Calculus is all about how things change, whether that be how something is changing at a given point or how much it has changed over a given time. It turns out that we can answer these questions with limits. A shorthand way to represent the limit as x approaches c of some function $f(x)$ is L. Namely,

$$\lim_{x \to \infty} f(x) = L$$

Note: it is **very** important that you understand the example problems in this chapter.

10.1 Refresher

To illustrate a basic example of a limit, consider the graph $f(x) = x^2$. If we want to find the limit as x approaches 2, this is simple! Because $f(x)$ is continuous, we simply plug in 2 to find the limit. Therefore,

$$\lim_{x \to 2} x^2 = 4$$

What about a function that is discontinuous at some point? Consider a limit of $f(x) = \frac{x-2}{x^2-4}$

$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

Plugging in 2 to $f(x)$, we see that the denominator is zero, so this simple method of plugging in won't work. However, we can cancel out terms. Factoring the denominator, we get

$$f(x) = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}$$

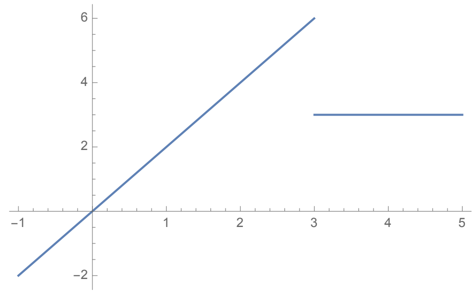
Notice how the function $\frac{1}{x+2}$ is just a continuous version of the function $\frac{x-2}{(x+2)(x-2)}$; it the exact same except that the hole at $x = 2$ is not present. Because of this, we can find the limit by plugging $x = 2$ into the continuous function. Doing this, we get the point $(2, \frac{1}{4})$, which is the coordinate of the hole in the function $f(x) = \frac{x-2}{x^2-4}$. Thus,

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$

Sometimes, however, the limit of a function does not exist. Consider the piecewise function

$$f(x) = \begin{cases} 2x & \text{if } x < 3 \\ 3 & \text{if } x > 3 \end{cases}$$

Looking at the graph, we can see jump a discontinuity at the point $x = 3$



Because of this, the limit *does not exist* because $f(x)$ does not approach the same value from both sides of $x = 3$. However, each side has a definite limit:

$$\lim_{x \to 3^-} f(x) = 6$$

$$\lim_{x \to 3^+} f(x) = 3$$

This is one way to describe a function that has a jump discontinuity. However, it is important to remember that the **limit does not exist.** Another case is when the limits are infinite. For example, consider the function $f(x) = \frac{1}{x}$. If we wish to compute $\lim_{x \to 0} f(x)$, the denominator would be zero which would result in an **infinite limit**. This would be the same outcome if we tried to compute the limits from each side. We can use this infinite limit to define a vertical asymptote. A function $f(x)$, when graphed, has a vertical asymptote at $x = c$ if any of the following conditions are met:

$$\lim_{x \to c-} f(x) = \infty \qquad \lim_{x \to c-} f(x) = -\infty \qquad \lim_{x \to c+} f(x) = \infty \qquad \lim_{x \to c+} f(x) = -\infty$$

Finally, we know that the horizontal asymptote of a function is defined as the y-value that the function approaches as x approaches $\pm\infty$. Namely, our horizontal asymptote $y = L$ of function $f(x)$ is defined as

$$\lim_{x \to -\infty} f(x) = L$$

OR

$$\lim_{x \to \infty} f(x) = L$$

10.2 Properties of Limits

The properties of limits are, to say it short, what you'd expect. Proving some of these would require knowledge about the epsilon-delta definition of a limit, which is part of the calculus curriculum. So, for now, just accept that the following identities are true. None of these properties should come as a surprise:

$$\lim_{x \to c} k = k \quad \text{if } k \text{ is a constant} \tag{81}$$

$$\lim_{x \to c} x = c \tag{82}$$

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) \tag{83}$$

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \tag{84}$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) \tag{85}$$

$$\lim_{x\to c}(f(x)\bullet g(x)) = \lim_{x\to c} f(x) \bullet \lim_{x\to c} g(x) \tag{86}$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \tag{87}$$

$$\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n \tag{88}$$

Once you study these identities for a few minutes, you'll realize that they really are intuitive; there are no weird tricks that you'll have to memorize for these.

Example 10.1

If $f(x) = 3x^2 - 2x$ and $g(x) = 2x+3$, find:

(a) $\lim_{x\to 2}(f(x)+g(x))$

(b) $\lim_{x\to 3}(f(x)-g(x))$

(c) $\lim_{x\to 1}(f(x)\bullet g(x))$

(d) $\lim_{x\to -1}\frac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}[f(x)\bullet g(x)]^3$

For part (a), we can use the sum identity above to get

$$\lim_{x\to 2}(f(x)+g(x)) = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)$$

Evaluating each limit,

$$\lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = 8+7 = \boxed{15}$$

For part (b), we use the subtraction identity:

$$\lim_{x\to 3}(f(x)-g(x)) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$$

$$= 21 - 9 = \boxed{12}$$

For part (c), we use the multiplicative identity:

$$\lim_{x\to 1}(f(x)\cdot g(x)) = \lim_{x\to 1} f(x)\cdot \lim_{x\to 1} g(x)$$

$$= 1\cdot 5 = \boxed{5}$$

For part (d),

$$\lim_{x\to -1}\frac{f(x)}{g(x)} = \frac{\lim_{x\to -1} f(x)}{\lim_{x\to -1} g(x)}$$

$$= \frac{5}{1} = \boxed{5}$$

For part (e), we can define a function $h(x) = f(x)\cdot g(x)$:

$$\lim_{x\to 2}[f(x)\cdot g(x)]^3 = \lim_{x\to 2}[h(x)]^3 = [\lim_{x\to 2} h(x)]^3$$

We can plug back in for $h(x)$ to get

$$[\lim_{x\to 2} f(x)\cdot g(x)]^3$$

Solving for $\lim_{x\to 2} f(x)\cdot g(x)$, we use the same method that we did before:

$$\lim_{x\to 2} f(x)\cdot g(x) = 8\cdot 7 = 56$$

Now, we cube 56 to get our final answer: $\boxed{56^3}$

If we are asked to find the limit of a composition of functions, fear not! This limit is a fairly simple one:

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) \tag{89}$$

However, there are some special cases we must worry about. One of these cases is if the limits of the functions do not exist. Namely, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist. One may be inclined to think that in an operation with both functions whose limits do not exist, the limit of the composition will not exist. However, this is not necessarily true. If we, for example, are trying to compute the limit $\lim_{x\to 3}(f(x)+g(x))$, where $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist, we quickly notice that this is not possible. We can take an alternate approach to this, though. If we take the limit from one side ($x\to 3^-$) and then take the limit of the other side ($x\to 3^+$), the limit is defined **if these two limits are equal.** Namely, if

$$\lim_{x\to 3^-}(f(x)+g(x)) = \lim_{x\to 3^+}(f(x)+g(x))$$

then the limit **exists**.

10.3 Derivatives

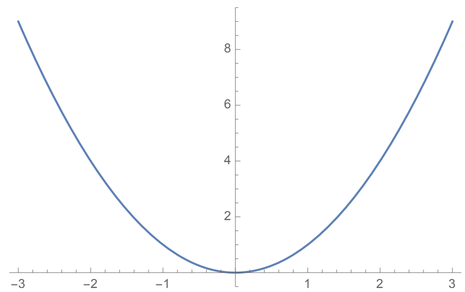
As you may recall from Algebra 1, we studied a way to represent the rate of change of a linear function. We described this as the *slope* of the line, or how much y changed for a movement of one unit to the right in the x direction. Namely,

$$\text{Slope} = \frac{\Delta y}{\Delta x} \tag{90}$$

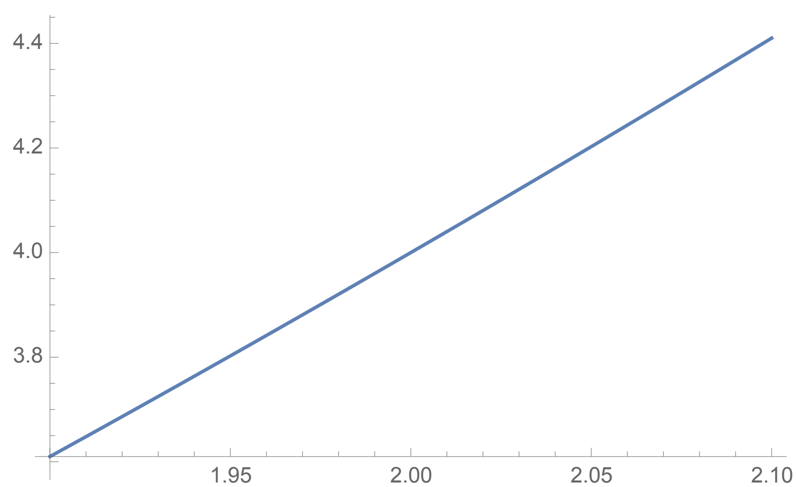
This formula can be slightly manipulated, however. Notice how our change in y, Δy, is just a change in the *output* of the functions whose inputs are $x+\Delta x$ and x, respectively. Because of this, we can rewrite Δy as $f(x+\Delta x)-f(x)$. Our change in x is $(x+\Delta x)-x=\Delta x$, so nothing changes for that. Rewriting our slope formula,

$$\text{Slope} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \tag{91}$$

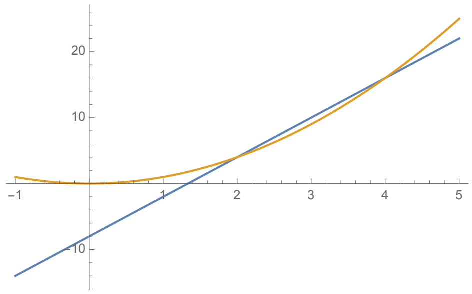
How do we represent the slope of a curve, though? Obviously, taking Δy and Δx at different intervals would not work. **The rate of change at a point is defined as the slope of the tangent line to that point in the graph**. The idea is that the smaller Δx gets, the more linear the shape of the curve becomes. Because of this, the smaller our Δx and Δy are, the more accurate our slope will be. For example, take a look at the graph $f(x)=x^2$:



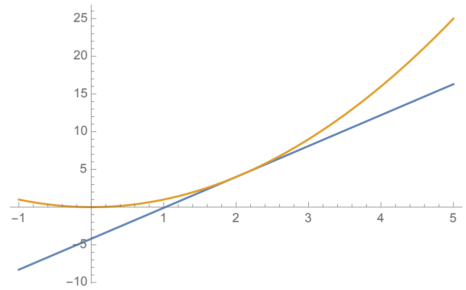
If we wish to compute the rate of change at the point $(2,4)$, we could take successively smaller Δx. Consider x^2, but one the x interval $1.9<x<2.1$:



Clearly, the graph is very close to a line. Now, we can take our calculators out, plug in a very small Δx, and find a number that is close to the rate of change (within a margin of error that we can calculate). Consider the following graphs. Both are of x^2, but one has a Δx of 3 when calculating the slope of the tangent line, and the other has a Δx of 0.1:



Tangent line: $y-4=6(x-2)$



Tangent line: $y-4=4.1(x-2)$

You may have realized by now that there is a deeper way to represent this. To get an exact rate of change for our curve, we must have an infinitely small Δx. As Δx gets smaller, $f(x+\Delta x)$ and $f(x)$ approach the same value as well. This would result in two extremely (infinitely) small numbers in both the numerator and the denominator, making it impractical to actually find the slope. However, there is a way around this: the limit. Namely, we take the limit as Δx approaches zero of our slope function:

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The result is a function, $f'(x)$, which takes an x-value as an input and gives the slope as its output. This is known as the **derivative**.

Example 10.2

Find the derivative of $f(x)=x^2$.

We know that $f(x+\Delta x)=(x+\Delta x)^2$, since we are simply changing the argument of our function:

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$$

Now, we simply have to evaluate the limit; no easy tricks here!

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta x^2-x^2}{\Delta x} &= \lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(\frac{2x\Delta x}{\Delta x}+\frac{\Delta x^2}{\Delta x}\right)\\ &= \lim_{\Delta x\to 0}(2x+\Delta x)\end{aligned}$$

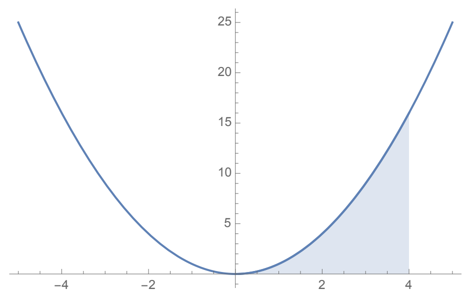
Now that we have all our Δx's out of the denominator, we don't need to worry about having an undefined expression. We can simply plug $\Delta x = 0$ in and solve!

$$f'(x) = 2x$$

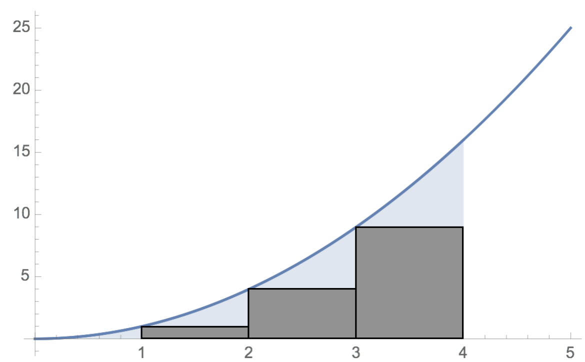
We have just proven that the derivative of x^2 is $2x$. This tells us everything we need to know about the rate of change of $f(x)$. When $x = 5$, the rate of change, or slope of the tangent line to x^2 at $x = 5$, is 10. When $x = -3$, the rate of change is -6.

10.4 Area Under a Curve

Consider the curve $f(x) = x^2$ again. What if we want to find the area under the curve from $x = 0$ to $x = 4$?



Clearly, there isn't any geometric trick we can use to find this area; the shape of the shaded area is something completely new, whose area simply doesn't fall into that of a common polygon's. We can estimate (and come pretty close to) the area of the shaded region by taking the area of rectangles under the curve:



The width of each rectangle is 1. The height of each rectangle, however, changes. It is $f(x)$, where x is the leftmost point on the base of the rectangle. Thus, summing all these areas up is called a **left-hand Riemann sum**. More formally, if we want to find the area under a curve with a Riemann sum of n rectangles over a distance Δx (in this case, our Δx is 4), then the width of each rectangle is $\frac{\Delta x}{n}$. For our problem, we would have four rectangles of width $\frac{4}{4} = 1$. Now, you may ask, "Why are there only three rectangles in the diagram?" This is because the height of the "first" rectangle is zero, since $f(0) = 0$. This results in the rectangle having zero area. Our area, thus, is the sum of the areas of these rectangles:

$$\begin{aligned} A^{\text{tot}} &= \sum A_i \\ &= f(0) * 1 + f(1) * 1 + f(2) * 1 + f(3) * 1 \\ &= f(1) + f(2) + f(3) \end{aligned}$$

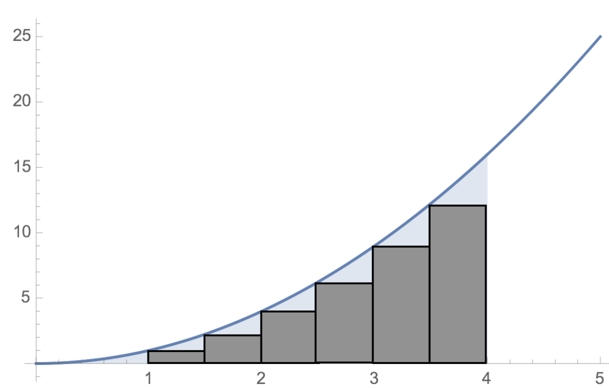
Now, we plug into x^2 to get

$$1 + 4 + 9 = \boxed{14}$$

Thus, our estimate of the area under the curve is 14. This is an underestimate, as the diagram clearly shows. The following statements are very important regarding left and right-hand Riemann sums:

1. If a function is increasing on an interval, then a left-hand sum on that interval will always be an underestimate.
2. If a function is increasing on an interval, then a right-hand sum on that interval will always be an overestimate.
3. If a function is decreasing on an interval, then a left-hand sum on that interval will always be an overestimate.
4. If a function is decreasing on an interval, then a right-hand sum on that interval will always be an underestimate.

Now, how do we get a more accurate area under our curve? We take the areas of more rectangles! Assuming our interval stays constant, the only way we can do this is to decrease the width of each rectangle. Going back to the x^2 example, we can see that if we increase the number of rectangles, the area increases and comes closer to the actual area:



If we change each width to $\frac{1}{2}$, the area would be

$$\frac{1}{2}\left[f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)+f(3)+f\left(\frac{7}{2}\right)\right]$$

Once we evaluate this sum, we get an area of $\frac{35}{2}$, or 17.5.
It is important to realize that both of these sums can be represented in sigma notation. The sum we just did is

$$\sum_{i=1}^{8}\frac{1}{2}f\left(\frac{i}{2}\right)=\sum_{i=1}^{8}\frac{1}{2}\left(\frac{i}{2}\right)^2$$

More generally, if we are finding the area under a curve $f(x)$ on the interval $x=a$ to $x=b$ with n rectangles, the total area is

$$\sum_{i=1}^{n}\frac{b-a}{n}f\left(a+i\frac{b-a}{n}\right)$$

Where $\frac{b-a}{n}$ is the width ($b-a$ is Δx). Since we can compute the width of each triangle easily, the sum simplifies to:

$$\sum_{i=1}^{n}wf\left(a+iw\right)$$

Where w is the width of each rectangle.
So, how do we find the exact area under the curve? Again, we must take a limit. In this case, we want the limit as the number of rectangles approaches infinity:

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}wf\left(a+iw\right)=\sum_{i=1}^{\infty}wf\left(a+iw\right) \tag{92}$$

This is known as a **definite integral**. Although we won't cover the properties of integrals in this, we denote the following infinite sum of $f(x)$ from $x=a$ to $x=b$ as

$$\sum_{i=1}^{\infty}wf\left(a+iw\right)=\int_a^b f(x)dx \tag{93}$$

Example 10.3

Let's return to our x^2 example. Find the exact area under the curve from $x=0$ to $x=4$.

Using the limit definition to find the area under the curve,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left(i\frac{4}{n}\right)^2$$

Now we have to utilize our knowledge of infinite sums to compute this. Since $\frac{4}{n}$ is a constant, we can pull it out to get

$$\lim_{n\to\infty}\left(\frac{4}{n}\sum_{i=1}^{n}\left(i\frac{4}{n}\right)^2\right)=\lim_{n\to\infty}\left(\frac{4}{n}\sum_{i=1}^{n}i^2\frac{16}{n^2}\right)$$

Pulling out the $\frac{16}{n^2}$,

$$=\lim_{n\to\infty}\left(\frac{64}{n^3}\sum_{i=1}^{n}i^2\right)$$

To get rid of the sigma, we use our sum of squares formula:

$$=\lim_{n\to\infty}\left(\frac{64}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$$

All we have to do now is solve the limit. We do this by expanding all the terms and separating the fraction. We can start, however, by simplifying the coefficients and cancelling out a factor of n:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{64}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)&=\lim_{n\to\infty}\left(\frac{32}{3n^2}\right)(n+1)(2n+1)\\&=\lim_{n\to\infty}\left(\frac{32}{3n^2}\right)(2n^2+3n+1)\\&=\frac{32}{3}\left[\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{n^2}\right)\right]\\&=\frac{32}{3}\left[\lim_{n\to\infty}\left(\frac{2n^2}{n^2}+\frac{3n}{n^2}+\frac{1}{n^2}\right)\right]\end{aligned}$$

Since $\lim\limits_{n\to\infty}\frac{3n}{n^2}$ and $\lim\limits_{n\to\infty}\frac{1}{n^2}$ are both 0 (see page 73), our limit then becomes

$$\frac{32}{3}\left[\lim_{n\to\infty}\left(\frac{2n^2}{n^2}\right)\right]$$

Evaluating, we get our **exact** area under the curve:

$$\boxed{\frac{64}{3}}$$

10.5 Problems

1. Evaluate the following limits:

   (a) $\lim\limits_{x \to 1} \dfrac{3x^3 + 2x^2 - 3x - 2}{(x+1)}$

   (b) $\lim\limits_{x \to -\frac{2}{3}^-} \dfrac{3x^3 + 2x^2 - 3x - 2}{(x+1)|x + \frac{2}{3}|}$

   (c) $\lim\limits_{\theta \to 0} \dfrac{\sec\theta - 1}{\theta}$

   (d) $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$

   (e) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{2x}$

   (f) $\lim\limits_{x \to 0} \left(1 + x\right)^{1/x}$

2. Find the derivatives of the following functions

   (a) $f(x) = \sqrt{3x}$

   (b) $f(x) = e^x$ Note: $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

   (c) $f(x) = 2x^3$

   (d) $f(x) = \frac{2}{x}$

   (e) ★ $f(x) = \ln x$ Note: $e = \lim\limits_{h \to 0} (1+h)^{1/h}$

3. The power rule is a fundamental property of functions that is used when taking their derivatives. The theorem states that if we have a function $f(x) = ax^b$, then the derivative for that function, $f'(x)$, is abx^{b-1}. Prove this is true.

4. Prove that the derivative for any constant function, $f(x) = c$, is **always** zero.

5. Estimate the area under the curve $y = x^2 + 2$ on the interval $[1,3]$ using a right-hand Riemann sum with four subintervals.

6. Estimate the area under the curve $y = e^x$ on the interval $[0, 2e]$ using a left-hand Riemann sum with four subintervals.

7. Find the area under the following curves on their specified intervals

   (a) $f(x) = 2x^3$, on the interval $(1,4)$

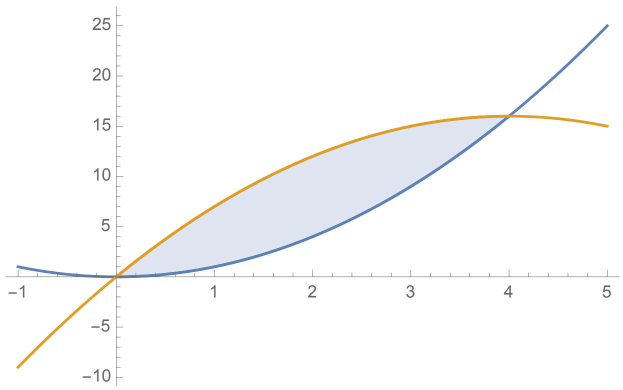
   (b) $f(x) = 3x - 2$, on the interval $(2,3)$

(c) $f(x) = \sqrt[3]{x}$, on the interval $(0,5)$

(d) $f(x) = \frac{1}{x}$, on the interval $(1,2)$

(e) $f(x) = 2^x$, on the interval $(3,5)$

8. The two graphs below show the functions $f(x) = x^2$ and $f(x) = -x^2 + 8x$. Find the area in the shaded region.



11 Miscellaneous Topics

This chapter will cover any topics that are part of the curriculum but do not neatly fit into any of the chapters before.

11.1 Partial Fraction Decomposition

Partial fractions is a way of representing a fraction in multiple parts. In Calculus, this becomes absolutely necessary if we want to integrate a rational fractions (you do not have to know what that means). For example, suppose we want to represent the fraction $\frac{2x+18}{x^2+x-6}$ in two parts. First, we would factor the denominator:

$$\frac{2x+16}{x^2+x-6} = \frac{2x+16}{(x+3)(x-2)}$$

Once we have the denominator neatly factored, we can state what we want to actually do with the expression. We want to split the fraction into two parts, where the denominator of each part is a factor of the overall denominator. Namely,

$$\frac{2x+18}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \qquad \text{Where } A \text{ and } B \text{ are constants.}$$

To find what A and B are, we want to first get rid of all our fractions; this makes it much easier. To do this, we will multiply both sides by the overall denominator:

$$(x+3)(x-2)\frac{2x+16}{(x+3)(x-2)} = (x+3)(x-2)\left(\frac{A}{x+3} + \frac{B}{x-2}\right)$$

Multiplying, we get

$$2x+16 = A(x-2)+B(x+3)$$

Now, finding A and B becomes easy. We just have to plug in values such that one of the terms on the right side of the equation becomes zero. If we want to find B, we have to find an x value such that $A(x-2)$ is zero. After (hopefully) a few seconds, we realize that x value is 2:

$$2(2)+16 = A(2-2)+B(2+3)$$

$$20 = 5B$$

$$B = 4$$

Now that we can find A. Given that we know the value of B, we can plug in any value (except 2) for x to find A. However, we will plug in $x = -3$ so that we don't have to worry about plugging in B at all:

$$2(-3)+16 = A(-3-2)+B(-3+3)$$

$$10 = -5A$$

$$A = -2$$

Now that we know A and B, we plug them back into our original equation, which is

$$\frac{2x+18}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$\frac{2x+18}{(x+3)(x-2)} = \frac{-2}{x+3} + \frac{4}{x-2}$$

In order to use partial fractions to simplify an expression, it is absolutely necessary for the degree of the numerator to be less than the denominator. If that is not true, then we must use polynomial division to reduce the fraction. For example, consider the fraction $\frac{2x^2+13x-22}{x^2+5x-6}$. Obviously, the degree of the numerator is the same as the denominator. Using polynomial division, we get

$$\begin{array}{r|r} & 2 \\ \hline x^2+5x-6 & 2x^2+13x-22 \\ & -2x^2-10x+12 \\ \hline & 3x-10 \end{array}$$

With a quotient of 2 and remainder of $3x-10$, we can setup our new expression:

$$\frac{2x^2+13x-22}{x^2+5x-6} = 2 + \frac{3x-10}{x^2+5x-6}$$

Now that we have the degree of the numerator less than the denominator, we can use partial fraction decomposition. Although we are putting the fraction $\frac{3x-10}{x^2+5x-6}$ into partial fractions, it is important to remember that we are also adding 2 to the fraction. However, when we are decomposing, we don't need to worry about it; we just have to plug it in at the end.

$$\frac{3x-10}{x^2+5x-6} = \frac{3x-10}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1}$$

$$(x+6)(x-1)\frac{3x-10}{(x+6)(x-1)} = (x+6)(x-1)\left(\frac{A}{x+6} + \frac{B}{x-1}\right)$$

$$3x-10 = A(x-1) + B(x+6)$$

Again, to find A and B, we plug in values for x such that one of the right terms becomes 0:

$$3(-6)-10 = A(-6-1) + B(-6+6)$$

$$-28 = -7A$$

$$A = 4$$

Plugging in 1 to solve for B,

$$3(1)-10 = A(1-1) + B(1+6)$$

$$-7 = 7B$$

$$B = -1$$

Now that we have both the values for A and B, we can plug them back into the original equation.

$$\frac{3x-10}{x^2+5x-6} = \frac{4}{x+6} + \frac{-1}{x-1}$$

We're not done yet, though! Remember, we are not trying to decompose $\frac{3x-10}{x^2+5x-6}$; rather, we are trying to decompose $\frac{2x^2+13x-22}{x^2+5x-6}$, which we already figured out was $2+\frac{3x-10}{x^2+5x-6}$. Substituting our partial fractions in for $\frac{3x-10}{x^2+5x-6}$, we get our final decomposition:

$$\frac{2x^2+13x-22}{x^2+5x-6} = 2 + \frac{4}{x+6} + \frac{-1}{x-1}$$

In short, remember the 2!

Our last form of decomposition is when we have a fraction where the numerator has a repeated linear factor; this means that the denominator has two or more identical factors. An example of this is the fraction $\frac{-2x^2-5x-1}{x(x+2)^2}$. When we decompose a fraction like this, we do not separate it into $\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+2}$, as you may expect. Rather, we separate it into three fractions, while increasing the power of the denominator for the fractions with the repeated linear factor. Namely,

$$\frac{-2x^2-5x-1}{x(x+2)^2} = \frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

Multiplying through, we get

$$-2x^2-5x-1 = A(x+2)^2 + B(x)(x+2) + C(x)$$

Plugging in $x = -2$ to solve for C,

$$-2(4)-5(-2)-1 = -2C$$

$$1 = -2C$$

$$C = -\frac{1}{2}$$

If we want to solve for A, we plug $x = 0$ in to make $B(x)(x+2)$ and $C(x)$ zero.

$$-2(0)^2-5(0)-1 = A(0+2)^2 + B(0)(0+2) + C(0)$$

$$-1 = 4A$$

$$A = -\frac{1}{4}$$

As you may have noticed, there is no way to simply isolate B; it is multiplied by factors that A and C are also multiplied by. However, since we know what A and C are, we can plug any value in for x that isn't -2 or 0; we will go with $x = 1$:

$$-2(1)^2 - 5(1) - 1 = A(1+2)^2 + B(1)(1+2) + C(1)$$
$$-8 = -\frac{1}{4}(9) + 3B - \frac{1}{2}$$
$$-\frac{21}{4} = 3B$$
$$B = -\frac{7}{4}$$

Plugging all values back into the original equation,

$$\frac{-2x^2 - 5x - 1}{x(x+2)^2} = -\frac{1}{4x} - \frac{7}{4(x+2)} - \frac{1}{2(x+2)^2}$$

Problems

1. Decompose the following expressions into partial fractions:

   (a) $\dfrac{1}{x^2 - 4}$

   (b) $\dfrac{x}{(x+2)^2(x+3)}$

   (c) $\dfrac{5x^2 + 3x - 2}{x^2 + 2x - 24}$

   (d) $\dfrac{2x^2}{(x+3)(x+4)}$

11.2 Projectile Motion Equations

The projectile motion equations give us the x and y-coordinates of any object that has been launched with a velocity v_0 at an angle θ to the ground **as a function of time**. For a projectile launched from the point (x_0, y_0), the equations are

$$x(t) = (v_0 \cos\theta)t + x_0$$
$$y(t) = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 + y_0$$

where g is the gravitational constant, $9.81\,{}^{\text{m}}/_{\text{s}^2}$ or $32.2\,{}^{\text{ft}}/_{\text{s}^2}$. The actual specifics as to why these equations work you will learn in physics and calculus, but for now just accept that these are true.

Example

A football is kicked with an initial speed of $50\frac{\text{m}}{\text{s}}$ at an angle of 30° with respect to the ground. 50 feet out, there is a field goal; the minimum height to make it through is 10 feet.

(a) Does the football make it through the field goal?

(b) By how many feet does the football make it/miss by?

Setting up our projectile motion equations, we choose the point where the ball is kicked from to be $(0, 0)$; you can choose anywhere to be your origin, but this is the neatest way to solve the problem.

$$x(t) = (50 \cos 30°)t$$
$$y(t) = (v_0 \sin 30°)t - \tfrac{1}{2}gt^2$$

What we essentially are trying to do is find the point in the football's path that gives us the most information about how it passes the field goal. This point is clearly $(50, y)$, because that is right when the ball is even with the field goal; the if $y > 10$, then the field goal is made. If $y < 10$, then the field goal is missed. To find what y is, we first have to find the time t that it crosses the field goal. To do this, we plug in $x = 50$:

$$50 = (50 \cos 30°)t$$

$$t = 1.155\text{s}$$

Now, in order to find y, we must plug in t to the $y(t)$ equation:

$$y = (50 \sin 30°)(1.155) - \frac{1}{2}g(1.155)^2$$

$$\boxed{y = 7.40\text{ft}}$$

Since the crossbar is 10 feet high, the football does not make it through the field goal; it misses it by $(10-7.40)$ft, or 2.60ft.

Problems

1. A rocket is launched with an initial horizontal velocity of $600\frac{m}{s}$ at an angle 77° off the ground.
   (a) What is the maximum height that the rocket achieves? When does it achieve this height?
   (b) What is the speed of the rocket at $t = 3s$? $5s$?
   (c) How far away from the launch point does the rocket land? Assume the Earth is flat.
2. ★ The kicker on a football team can give the ball an initial speed of $25.0\frac{m}{s}$. Within what angular range must he kick the ball if he is to score a field goal from a point 50.0m in front of the goalposts whose horizontal bar is 3.44m above the ground?

11.3 Mathematical Induction

Mathematical induction states that if we hypothesize a statement is true, and if we prove that it holds true for a concrete number and arbitrary numbers n and $n+1$, then the statement is true for any number above that base number. This idea is derived off of Peano's axioms, which essentially state that every natural number can be obtained by counting from zero upwards, due to the fact that no counting is needed to obtain zero. Mathematical induction states that if we have a function $P(n)$ where $n \in \mathbb{N}$, and we can prove that $P(n)$ is true for $n = n_0$, then if we assume $P(n)$ is true for $n > n_0$ and prove it is true for $n+1$, $P(n)$ is true. Although this may seem confusing, let's look at an example:

Example 11.3.1

Prove that $P(n) = 1+2+3+...+n = \frac{n(n+1)}{2}$

First, let's prove that this statement is true for $n = 1$:

$$1 = \frac{1(1+1)}{2} = 1$$

Now, we must *assume* that the statement is true for $n = k$, where $k > 1$. Namely,

$$1+2+3+...+k = \frac{k(k+1)}{2}$$

If we assume the statement is true for $n = k$, then we can prove it is true for $n = k+1$:

$$1+2+3+...+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

We know, however, that $1+2+3+...+k = \frac{k(k+1)}{2}$, based off our original assumption. Because of this, we can substitute in $\frac{k(k+1)}{2}$:

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

Getting rid of the denominator,

$$k(k+1) + 2(k+1) = (k+1)(k+2)$$

Then, once we expand the left side and factor, we get1

$$(k+1)(k+2) = (k+1)(k+2)$$

We have proven that $P(n) = 1+2+3+...+n = \frac{n(n+1)}{2}$ is true based on the fact that $P(1)$ is true and $P(k+1)$ is true *if* $P(k)$ is true. If we let $k = 1$, then $P(2)$ must be true, since 2 is just $1+1$. Since $P(2)$ is true, then $P(3)$ must be true, since $3 = 2+1$... the chain continues.

Problems

1. Prove that $\frac{1}{1(2)} + \frac{1}{2(3)} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$
2. Prove that $\frac{3}{(1)(2)(2)} + \frac{4}{(2)(3)(2^2)} + \frac{5}{(3)(4)(2^3)} + ... + \frac{n+2}{(n)(n+1)(2^n)} = 1 - \frac{1}{(n+1)(2^n)}$
3. Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 while $n > 0$.
4. Prove that $4n^3 + 8n$ is divisible by 6 while $n > 0$
5. ★ Prove the Binomial Theorem (covered in next section):

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j$$

11.4 Binomial Theorem

Binomial theorem states that if we have a binomial $(a+b)^n$, it can be expanded into a series of terms:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n = \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j \quad (94)$$

Recall that operator $\binom{n}{j}$ is equal to $\frac{n!}{j!(n-j)!}$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} \quad (95)$$

Where

$$n! = n(n-1)(n-2)(n-3)(...)(3)(2)(1) \quad (96)$$

The proof for this is long and and rigorous, but readily available online. In short, mathematical induction is used to prove this statement is true. Although the theorem itself may be daunting, all it takes is a few examples to get a good grasp of the idea.

Example

Given that the a term in an expanded binomial is $\binom{7}{3}a^4 * b^3$,

(a) find the binomial in the form $(a+b)^n$.

(b) find what number term this is in the expanded binomial.

Looking at the formula, we can see that $n = 7$; it is the sum of the exponents. So, we can write the binomial as

$$(a+b)^7$$

Additionally, we know that the expanded binomial will start with $\binom{7}{0}a^7$ and then proceed with $\binom{7}{1}a^6b$, $\binom{7}{2}a^5b^2$... If we write out all the terms, we find that $\binom{7}{4}a^4b^3$ **is the fourth term**.

Problems

1. Expand the following binomials:

   (a) $(2x+y)^5$

   (b) $(\frac{2}{x}+y)^5$

   (c) $(3x-2y)^4$

2. What is the coefficient of the x^2 term in the expansion of $\left(\frac{1}{2}(x+x^2)\right)^8$?

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